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SUMMARY:
A finite element computer program for determining the deflection and stresses in general laminated composite plates was developed. The program is based on first-order shear deformation theory of laminated plates with small deflections. The program includes the flexibility to impose general boundary conditions, has three different quadrilateral elements, and is able to analyze both static and dynamic plate problems. The program code tested favorably against results from other analytical methods and literature sources. The results from this program demonstrate the need to include shear deformation in the analysis of composite plates.

PRIMARY SOURCES:

1. Agarwal, B. D. and Broutman, L. J. *Analysis and Performance of Fiber Composites,* 2nd Ed., John Wiley & Sons, New York (1990).

2. Reddy, J. N. *Energy and Variational Methods in Applied Mechanics: With an Introduction to the Finite Element Method,* John Wiley and Sons, New York (1984).

3. Reddy, J. N. *Introduction to the Finite Element Method,* McGraw-Hill, New York (1984).

4. Whitney, J. M. *Structural Analysis of Laminated Anisotropic Plates,* Technomic, Lancaster, PA (1987).

DEVELOPMENT OF A FINITE ELEMENT METHOD PROGRAM

FOR THE ANALYSIS OF LAMINATED COMPOSITE PLATES

USING FIRST-ORDER SHEAR DEFORMATION THEORY

A Thesis

Presented in Partial Fulfillment of the Requirements for

the degree Master of Science in the

Graduate School of The Ohio State University

by

Brett Arnold Pauer

* * * * *

The Ohio State University

1993

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FIELDS OF STUDY

Major Field: Mechanical Engineering

Studies in the Mechanics of Composite Materials, the Finite Element Method, Stress Analysis, and Fracture Mechanics.

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INTRODUCTION

Introduction

Composite materials have become widely used in engineering applications in the past couple of decades. This class of materials holds many benefits when used appropriately in engineering applications. Because of analysis uncertainties many composite components are "over-engineered" and the design is often governed by reiterative component testing. In these cases, the full benefit of composite materials is not realized. This has led to the development of analysis aids for several different structural member types.

One of the major composite structural members is the composite plate. A plate is a load carrying member which is bounded by two parallel planes called faces. Each face has the same characteristic length and width dimensions and are bounded by the plate edges. The distance between these faces is the plate thickness and this thickness is considered to be small compared to the dimensions of the faces. The plate faces can take on many different types of shapes (rectangular, circular, elliptical and others). Composite plates have been used in aeronautical structures for years. Composite plates are currently being used in land-based construction because of their exceptional environmental properties.

There are currently analysis programs which include composite materials. In addition to specific composite programs, several general finite-element programs available on large systems incorporate the analysis of composite materials. Analysis techniques for composites have been changing rapidly. Since these larger programs have included composites as an auxiliary component, they do not always keep up with current research in this area. Also, the size of these programs prohibit their use on microcomputers. Some authors have published computer programs for specific composite structures or limited composite material lay-ups. No author, however, has published a computer program for the analysis of general composite plates.

Objectives

The main objective of this research is to produce a working computer program for the analysis of general composite plates to be used on microcomputers. The program presented in this paper is limited to the analysis of laminated composite plates with elastic behavior and small deflections. Shear deformation is included in the analysis because of the material behavior response specific to composites. This program is an revision of an existing program published by J. N. Reddy [25]. Reddy's program was developed to analyze orthotropic materials with elastic behavior and small deflections. Although single-layered composites exhibit this behavior, most of the composite plates used in applications have more than one layer and require a more complex program for analysis.

To validate the computer code, results from the program are compared against results from other analytical methods and results from other authors in the literature. This test is used to insure that the program properly employs first-order shear

deformation theory (FSDT) of composite plates. Additionally, the effect of shear deformation in composite plates is observed by comparing the program results against results from a plate theory that does not include shear deformation. Because comprehensive instructions and documented source code are included, the program should prove to be a valuable educational aid for teaching the application of the finite element method to composite structures in advanced composite classes.

Overview

Chapter II begins with an introduction to composite materials with a background on their mechanics. An introduction to current composite plate theories ends the chapter. Definitions of variables and sign conventions used in the program and the rest of the paper are presented to aid in the reader's comprehension. Chapter III provides an introduction and derivation of first-order shear deformation theory of composite plates using variational energy formulation. Chapter IV shows how this theory is transformed into a finite element model for use in the computer program. Numerical results from the computer program are compared against other analytical method solutions to validate the program code in Chapter V. Finally, conclusions derived from the results and recommendations for future work are presented in Chapter VI.

It is assumed that the reader has a general knowledge of composite materials, plate theory, and the finite element and variational methods. Some background is presented in these areas to define terms and conventions used in the plate theory. For further information in these areas, see the following references: composite materials [1,30], plate theory [27,29,33], finite element and variational methods [8,10,21,24,25].

CHAPTER II

MECHANICS OF COMPOSITE MATERIALS AND PLATE THEORIES

Composite Materials

There are many types of composite materials used in the fabrication of structural components. The term "composite" refers simply to a material made of more than one distinct constituent. Composites have become known as materials which have clear boundaries between its constituents, and whose constituents have markedly different material properties. The constituents combine to form a composite material with material properties considerably different from any of its constituents. Most of the modern composites contain either particulates or fibers as main constituents. Particle-reinforced composites are formed by suspending either random or preferred orientation particles in a surrounding material. The material properties of these types of composites are obtained from load tests and are similar to isotropic (for random-oriented particulates) or orthotropic (for preferred orientation particulates) materials. Fiber-reinforced composites are made of fibers suspended in a surrounding material. The fibers may be either continuous or discontinuous (short-fiber). Fiber-reinforced composites may be either single-layered (including multiple plies of the same fiber orientation) or multi-layered. See Figure 1 for an outline of composite classifications. The program presented in this paper, COMPLATE, is useful for analyzing all of the above composite types. The most general case of composites are multi-layered continuous-fiber-reinforced hybrid composites. The mechanical

response of the other composite types can be modeled with simplifications to this general case. Also, continuous-fiber-reinforced composites are the most commonly used composites for structural components where high strength is required. For these reasons, the mechanics of continuous-fiber-reinforced composites are defined further and are utilized in the development of the computer program.

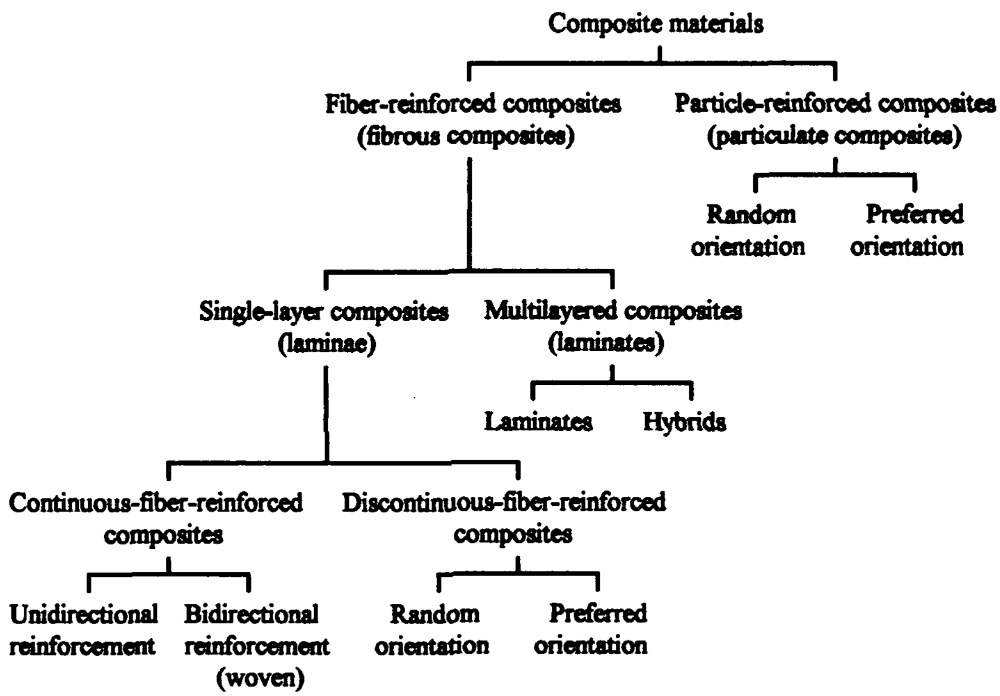

Figure 1: Types of Composite Materials [1].

Continuous-fiber-reinforced composites (hereafter, simply composites) differ in many ways from isotropic materials. Composites are generally composed of two distinct materials: reinforcements (fibers) and matrix (bonding material). Reinforcements made of fibers form the strength of a composite because they carry a majority of the load. Matrix is the material in between these fibers that binds the fibers and provides for load transfer between fibers in case of fiber breakage. The

matrix also protects the fibers from environmental degradation and damage due to handling. The matrix material generally has strength and stiffness properties much less than the reinforcement or fiber.

Composites often achieve strength-to-weight ratios significantly higher than metals. Atomic theory predicts strengths much higher than those actually found in practice for all materials. The reason for this shortfall in strength arises from inherent defects at both the microscopic (atomic) and macroscopic (visible) levels created during material processing. The largest allowable defect size at the macroscopic level depends on the cross-sectional area of the material. For bulk materials, relatively large defects can occur during material processing. For the manufacture of composite fibers, the size of defects is reduced because the cross-sectional area of the fiber is relatively small. If a visible defect is present in the fiber material, it breaks as it is stretched during manufacture. The unbroken portion of fibers have defect sizes limited to the microscopic level. By themselves, fibers are not useful for structural applications because of their small size and strength. A large number of fibers are bonded together by use of a matrix to form a high-strength material. There are many methods for manufacturing composite materials. The main concern of this paper is composite plate applications, so the following discussion refers to the structure of composite plates. However, for more information pertaining to the manufacturing of composites see references by Agarwal and Broutman [1], and Vinson and Sierakowski [30].

Lamina and Laminates

The ply is the basic building block of composite plates. A ply is the thin sheet of unidirectional fibers bonded by matrix material developed during manufacture. The ply is often many fiber diameters thick. A lamina or layer is formed when a unidirectional ply or combination of unidirectional plies of the same material with the same global fiber orientation is suspended in a matrix. Although it may consist of several plies, the important aspect of the lamina is that it is defined as a layer of material with common directional material properties. A multi-plied lamina contains a fiberless interface between plies which is relatively thin and is often ignored for analysis purposes. In practice, fibers are not equally spaced, but for schematic purposes, the lamina is often depicted having a single layer of fibers with universal fiber spacing as in Figure 2.

The material properties of composites differ from isotropic materials in the following way. Each lamina exhibits a generalized orthotropic behavior whose properties are different on three mutually perpendicular planes aligned with the fiber direction shown as 1, 2, 3 in Figure 2. Material properties are defined in the three directions corresponding to these planes.

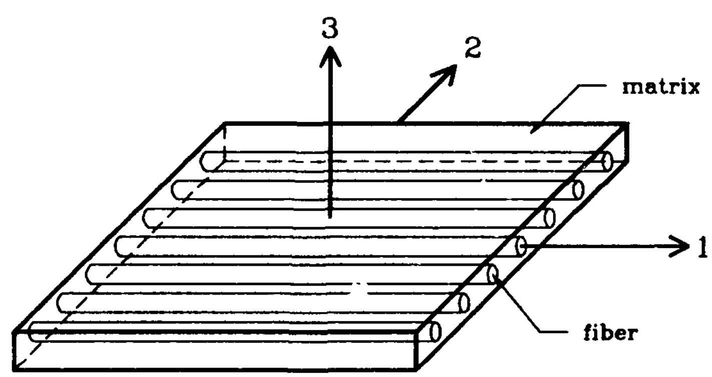

Figure 2: Schematic view of a lamina

These lamina are then stacked with the fibers aligned at different angles to form what is called a laminate as shown in Figure 3. The lamina are labeled according to their fiber angle relative to a global direction (x-axis). A code has been developed to label laminate stacking sequences. For example, [0/45/90] is a laminate composed of three equally thick lamina whose fibers are oriented 0°, 45°, and 90° respectively to the principle reference direction starting with the bottom layer (as shown in Figure 3). A subscript s, $[90/45/0]_S$, denotes a symmetric lay-up where the top layers are stacked in reverse order or [90/45/0/0/45/90] and a numerical subscript denotes the number of repeated plies, $[90_2/45_4/0_2]$ = [90/90/45/45/45/45/0/0] for example.

As the lamina are stacked to form a laminate, effective macroscopic properties are developed to characterize the laminate. These properties are assumed to be homogeneous although direction dependent (anisotropic) and are a weighted average of the properties of the composite constituents. Therefore, two laminates made of the same fiber and matrix material may have very different macroscopic material properties because of a difference in their stacking sequence.

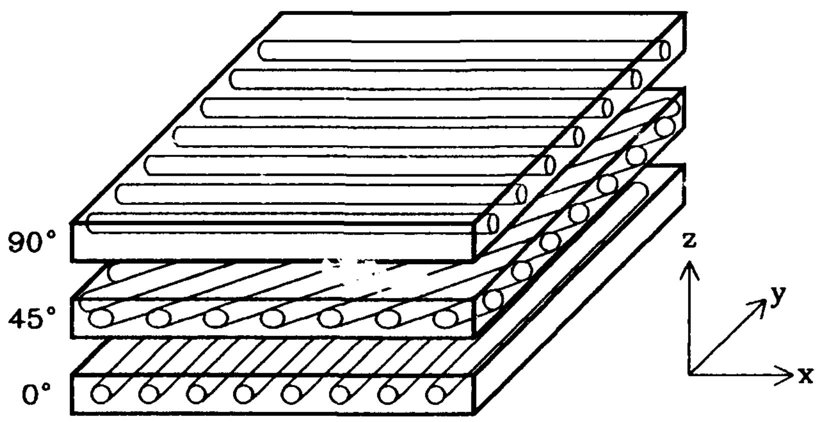

Figure 3: Schematic of a three-layered laminate [0/45/90].

General strengths and properties are experimentally determined for unidirectional lamina of composite materials. These lamina strengths are incorporated through the use of equations to predict effective macroscopic properties. Strengths of laminate stacking sequences are determined by one of several failure theories [1]. Test specimen are used to experimentally measure material properties and consist of small strips of composite. These specimen are checked for apparent flaws or defects and their edges are smoothed. The test specimen, therefore, form an ideal base-line on the strength of the composite.

Lamina Constitutive Relations

For a given lamina, the stiffness properties are generally given with respect to principal fiber directions. Direction-1 is aligned with the longitudinal direction of the fibers. Direction-2 is aligned with the direction transverse to the fibers in the lamina plane. Direction-3 is normal to both the 1 and 2 directions. Figure 4 shows these directions with respect to fiber alignment.

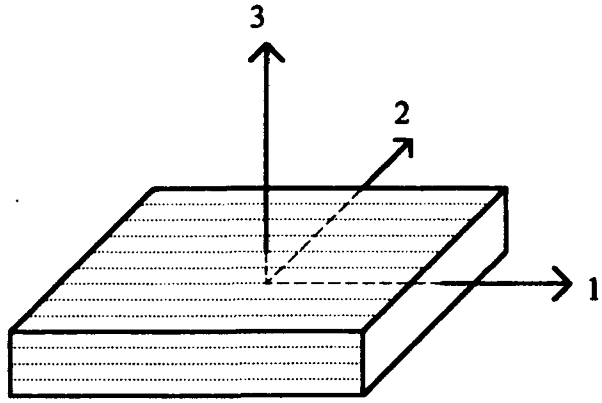

Figure 4: Principal fiber directions (1,2,3).

The material properties are determined through experimentation. Most lamina are characterized by the following independent material properties: E_1, E_2, ν_{12}, G_{12}, G_{23}, G_{13}. These properties are used to develop the stiffness matrix.

A general 6 x 6 orthotropic stiffness matrix relates the 6 principal normal and shear strains to the corresponding principal stresses [1]. For the laminate plate theory presented in this paper, the out-of-plane normal strain, ε_z, is assumed to be zero. This strain is uncoupled from the other strains and it allows the stiffness matrix to be reduced to 5 x 5. For each lamina, the orthotropic stiffness matrix aligned with principal fiber directions is defined by the following stress-strain relationship given in equations 2-1 and 2-2.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix} \tag{2-1}$$

where the matrix terms are defined as:

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} \qquad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} \qquad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$$
$$Q_{33} = G_{12} \qquad Q_{44} = G_{23} \qquad Q_{55} = G_{13} \tag{2-2}$$
$$\text{and } \nu_{21} = \frac{\nu_{12}E_2}{E_1}$$

The stiffness matrix given above is most useful for characterizing lamina properties. Laminates are formed by stacking layers of lamina with varying fiber orientations, thicknesses, and materials. To accommodate the variance in fiber orientation, the lamina stiffness matrices must be transformed to a common global orientation. For each lamina the fiber orientation is defined by the angle, θ_k, that the fiber direction makes with the x-axis. The angle θ_k is defined as positive in the counterclockwise direction and negative in clockwise direction as shown in Figure 5. The angle θ_k can have a value between 90° and -90°.

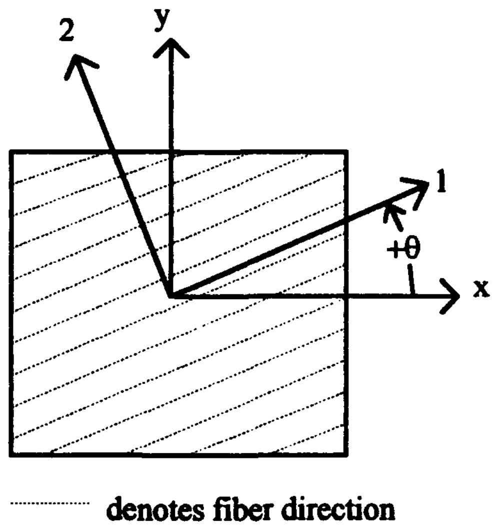

Figure 5: Global fiber orientation, θ.

The following relationships are based on a laminate of N layers. For each lamina (k = 1,2,3,...,N), the transformed stiffness matrix, $[\overline{Q}]$, is defined by the stress-strain relationship aligned with the global axes (x,y,z) as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{23} & 0 & 0 \\ \overline{Q}_{13} & \overline{Q}_{23} & \overline{Q}_{33} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} \tag{2-3}$$

where for each lamina, k:

$$\begin{aligned}
\overline{Q}_{11} &= Q_{11}m^4 + 2(Q_{12} + 2Q_{33})m^2n^2 + Q_{22}n^4 \\
\overline{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{33})m^2n^2 + Q_{12}(m^4 + n^4) \\
\overline{Q}_{13} &= Q_{11}m^3n - Q_{22}mn^3 - (Q_{12} + 2Q_{33})mn(m^2 - n^2) \\
\overline{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{33})m^2n^2 + Q_{22}m^4 \\
\overline{Q}_{23} &= Q_{11}mn^3 - Q_{22}m^3n + (Q_{12} + 2Q_{33})mn(m^2 - n^2) \\
\overline{Q}_{33} &= (Q_{11} + Q_{22} - 2Q_{12})m^2n^2 + Q_{33}(m^2 - n^2)^2 \\
\overline{Q}_{44} &= Q_{44}m^2 + Q_{55}n^2 \\
\overline{Q}_{45} &= (Q_{55} - Q_{44})mn \\
\overline{Q}_{55} &= Q_{44}n^2 + Q_{55}m^2
\end{aligned} \tag{2-4}$$

and $m = \cos\theta_k$, $n = \sin\theta_k$.

Laminate Constitutive Relations

In order to develop constitutive relationships that are independent of z, it is useful to define load and moment resultants. These resultants are the loads and moments per unit length along the lamina x and y cross-sections and acting through the laminate mid-plane. The orientation and positive direction are depicted in Figure 6.

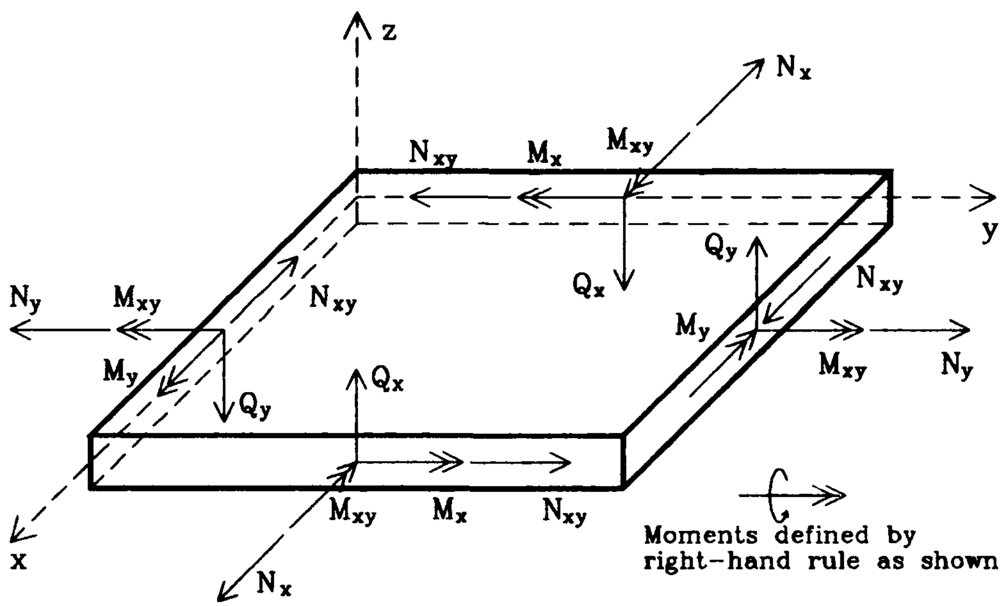

Figure 6: Orientation and positive direction of load and moment resultants.

The in-plane load resultants (N_x, N_y, N_{xy}) are defined as the integrals of the in-plane stresses through the thickness in the respective directions shown above in Figure 6.

$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x dz \qquad N_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y dz \qquad N_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} dz \tag{2-5}$$

The moment resultants (M_x , M_y , M_{xy}) are defined as the integrals of the moments created about the laminate mid-plane by the in-plane stresses through the thickness in the respective directions shown above in Figure 6.

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z dz \qquad M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z dz \qquad M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} z dz \tag{2-6}$$

The shear load resultants (Q_x and Q_y) are defined as the integrals of the transverse shear stresses through the laminate thickness.

$$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} dz \qquad Q_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} dz \tag{2-7}$$

Figure 7 shows the laminate coordinate system through the laminate thickness with terms used in the following sets of equations.

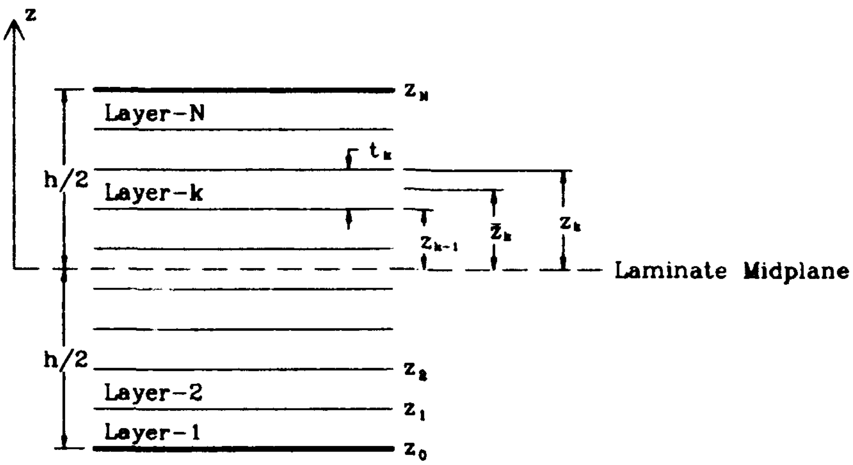

Figure 7: Laminate coordinates and terms.

The load resultant equations 2-5 to 2-7 along with the stress-strain relations 2-3 and 2-4 are used to develop the laminate stiffness matrices. The extensional stiffness, coupling stiffness, and bending stiffness matrices, A_{ij}, B_{ij}, D_{ij}, respectively, are defined by the following matrix equations:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ \hline M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ A_{12} & A_{22} & A_{23} & B_{12} & B_{22} & B_{23} \\ A_{13} & A_{23} & A_{33} & B_{13} & B_{23} & B_{33} \\ \hline B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} \\ B_{12} & B_{22} & B_{23} & D_{12} & D_{22} & D_{23} \\ B_{13} & B_{23} & B_{33} & D_{13} & D_{23} & D_{33} \end{array}\right] \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{2-8}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{avg} \\ \gamma_{xz}^{avg} \end{Bmatrix} \tag{2-9}$$

where $\varepsilon_x^o, \varepsilon_y^o, \gamma_{xy}^o$ are the in-plane strains at the laminate mid-plane, $\kappa_x, \kappa_y, \kappa_{xy}$ are the laminate curvatures, and $\gamma_{yz}^{avg}, \gamma_{xz}^{avg}$ are the average transverse shear strains as defined in the next chapter by first-order shear deformation theory.

The definitions of A, B, D plate stiffness matrix terms (i, j = 1,2,3) are given followed by simplifications for laminated plates.

$$
\begin{aligned}
A_{ij} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \overline{Q}_{ij}^{(k)} dz = \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)} (z_k - z_{k-1}) = \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)} t_k \\
B_{ij} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \overline{Q}_{ij}^{(k)} z dz = \frac{1}{2} \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)} (z_k^2 - z_{k-1}^2) = \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)} t_k \overline{z}_k \qquad \text{(2-10)} \\
D_{ij} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \overline{Q}_{ij}^{(k)} z^2 dz = \frac{1}{3} \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3) = \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)} \left(t_k \overline{z}_k^2 + \frac{t_k^3}{12} \right)
\end{aligned}
$$

The definitions of the terms z, z_k, z_{k-1}, t_k and $\overline{z}_k$ are given in Figure 7. The transverse shear matrix terms $A_{ij}(i, j = 4, 5)$ are given by the following equations.

$$
A_{ij} = k_{sc} \int_{-\frac{h}{2}}^{\frac{h}{2}} \overline{Q}_{ij}^{(k)} dz = k_{sc} \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)} \left[z_k - z_{k-1} \right] = k_{sc} \sum_{k=1}^{N} \overline{Q}_{ij}^{(k)} t_k \qquad \text{(2-11)}
$$

where k_{SC} is the shear correction factor. Methods for determining the value of the shear correction factor are presented in [5,31,33,34]

Inertial properties of the composite plate are required for dynamic analysis. The density of each lamina is given by $\rho^{(k)}$ where k is the layer number. The following terms represent transverse, transverse-rotation coupling, and rotational inertial properties respectively.

$$\rho_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho^{(k)} dz = \sum_{k=1}^{N} \rho^{(k)} (z_k - z_{k-1}) = \sum_{k=1}^{N} \rho^{(k)} t_k$$

$$\rho_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho^{(k)} z dz = \frac{1}{2} \sum_{k=1}^{N} \rho^{(k)} (z_k^2 - z_{k-1}^2) = \sum_{k=1}^{N} \rho^{(k)} t_k \bar{z}_k \qquad (2\text{-}12)$$

$$\rho_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho^{(k)} z^2 dz = \frac{1}{3} \sum_{k=1}^{N} \rho^{(k)} (z_k^3 - z_{k-1}^3) = \sum_{k=1}^{N} \rho^{(k)} \left(t_k \bar{z}_k^2 + \frac{t_k^3}{12} \right)$$

The relationships for general orthotropic-layered laminated plates given above can be applied to many other types of composite plates. Single-layered laminates can be accommodated by using a single layer (N=1) in the above equations. For preferred orientation (both fibrous and particulate) and bi-directional lamina, the orthotropic relationships along with the preferred orientation (θ) can be applied to a single layer. For random orientation lamina (discontinuous-fiber and particulate), isotropic relationships are obtained by using a single modulus of elasticity ($E_1 = E_2 = E$) a single shear modulus ($G_{12} = G_{23} = G_{13} = G$), and Poisson's ratio ($\nu_{12} = \nu$) in the above relationships.

Plate Theories

Plate theories are simplifications to general three-dimensional elasticity theory and were developed to analyze one of the basic structural member types, the plate. Three-dimensional elasticity theory may theoretically be used to analyze any solid object. Elasticity theory is often prohibitive in practical use because of the complexity of the solid object and the cost of applying the general theory to each analysis case.

Because of the general shape and load conditions common to plates, certain assumptions are made which reduce the equations of elasticity theory to several less-complex governing equations. There are two important points to remember when applying plate theories. First, the assumptions made in simplifying the governing equations limit the types of cases where a particular plate theory can be effectively applied. Second, the assumptions create an approximate solution for the plate problem. Other methods including testing should be used to determine the accuracy of the plate theory solution. Many of the plate theories were first developed for isotropic materials and were later adapted to composite materials.

Two-dimensional plate theories are developed by assuming displacement functions. These functions are characterized by equation 2-13. The displacement of any point in the plate (u_1, u_2, u_3) is defined by its mid-plane displacement (u,v,w), a function of the mid-plane coordinate (x,y), and an assumed form of the displacement through the plate thickness (U,V,W), dependent on the laminate mid-plane coordinate (x,y) and distance from the mid-plane (z). These functions are also dependent on time (t) for the case of dynamic problems. By separating the displacement functions into these two parts, the analysis can be reduced from three dimensions (x,y,z) to two dimensions (x,y).

$$\begin{aligned} u_1(x,y,z,t) &= u(x,y,t) + U(x,y,z,t) \\ u_2(x,y,z,t) &= v(x,y,t) + V(x,y,z,t) \\ u_3(x,y,z,t) &= w(x,y,t) + W(x,y,z,t) \end{aligned} \tag{2-13}$$

Classical plate theory (CPT) was the first form of the plate theories and is attributed to Kirkhhoff [20]. This theory was the first attempt to characterize thin

isotropic plates and is limited in its scope of applications due to many assumptions. The adaptation of this theory to laminated composite materials is generally attributed to Yang, Norris and Stavsky with additions by Whitney and Pagano [33]. The following assumptions provide the basis for CPT [35].

1. Plane sections of the plate cross section remain plane and normal to the mid-surface.
2. The deflections are small compared to the plate thickness.
3. Transverse normal strain is zero and transverse shear strains are negligible.
4. Transverse normal stress is negligible.

These assumptions result in the simplified displacement functions given in equation 2-14. These functions have three degrees of freedom (u,v,w) which are dependent on x, y, t only.

$$
\begin{aligned}
u_1(x,y,z,t) &= u(x,y,t) - z\frac{\partial w}{\partial x}(x,y,t) \\
u_2(x,y,z,t) &= v(x,y,t) - z\frac{\partial w}{\partial y}(x,y,t) \qquad (2\text{-}14) \\
u_3(x,y,z,t) &= w(x,y,t)
\end{aligned}
$$

This theory is adequate for a large class of isotropic plates and very thin composite plates. For thicker plates, with length and width to thickness ratios of less than 10, CPT tends to under predict the deflections in plate. This is caused from the transverse shear strains being larger than the assumption requires. This problem is

also encountered in moderately thin composite plates because of the directional material properties unique to composites.

One theory that addresses this inadequacy is the first-order shear deformation theory (FSDT). This theory was developed by Reissner for static analysis and refined by Mindlin for dynamic analysis of isotropic plates [18]. Yang et al. modified this theory for composite plates with further refinements by Whitney and Pagano [33]. This theory includes transverse shear strain in the analysis and gives better results for deflections and stresses in composite plates. Equation 2-15 shows the assumed displacement functions for this theory. Note that this theory allows five degrees of freedom $(u, v, w, \psi_x, \psi_y)$. This theory is known as first-order because the total displacements are assumed to be linear functions of z through the plate thickness. FSDT is presented in greater detail in the next chapter.

$$
\begin{aligned}
u_1(x,y,z,t) &= u(x,y,t) + z\psi_x(x,y,t) \\
u_2(x,y,z,t) &= v(x,y,t) + z\psi_y(x,y,t) \qquad (2\text{-}15) \\
u_3(x,y,z,t) &= w(x,y,t)
\end{aligned}
$$

FSDT does not adequately address boundary conditions on the plate faces or predict the interlaminar shear stresses through the plate thickness. In response, several higher-order shear deformation theories (HSDT's) have been presented based on work by Reissner and Schmidt for isotropic plates [18]. HSDT has been extended to laminated plates by Nelson and Lorch, Librescu, Lo et al. and Reddy [20]. These higher-order theories retain higher-order terms of z in the displacement function (see equation 3-1) than in CPT or FSDT. Several HSDT's have been presented by a number of authors. A sample of these theories can be found the following references

[3,6,12,13,14,18,19,22,26,28,35]. Equation 2-16 shows the general form of HSDT's. The number of degrees of freedom depends on the order of the displacement functions in terms of z and the assumptions of the particular higher-order theory.

$$
\begin{aligned}
u_1(x,y,z,t) &= u(x,y,t) + z\psi_x(x,y,t) + z^2\zeta_x(x,y,t) + z^3\phi_x(x,y,t) + \cdots \\
u_2(x,y,z,t) &= v(x,y,t) + z\psi_y(x,y,t) + z^2\zeta_y(x,y,t) + z^3\phi_y(x,y,t) + \cdots \\
u_3(x,y,z,t) &= w(x,y,t) + z\psi_z(x,y,t) + z^2\zeta_z(x,y,t) + \cdots
\end{aligned}
\tag{2-16}
$$

HSDT's produce more accurate transverse shear stress results than the two previous theories, however the resulting deflection and normal stresses show little improvement over FSDT [22]. They also require large mathematical and programming costs. Developing a program for one of these theories for a microcomputer is prohibited by the current computing capacity of existing microcomputers. FSDT is chosen for use in the computer program for its relatively small computing requirements balanced with its improved analysis results.

CHAPTER III

FIRST-ORDER SHEAR DEFORMATION THEORY OF COMPOSITE PLATES

General theory

The first-order shear deformation theory (FSDT) presented in this chapter is used for the revision of the computer program (COMPLATE) from the previously written program PLATE by J. N. Reddy [25]. This theory models general laminated composite plates and includes dynamic considerations. The main purpose of utilizing this theory is to transform a three-dimensional elasticity problem into a two-dimensional problem. The energy formulation is used to generate mass and stiffness matrices for application in the finite element method presented in the next chapter.

The need for this theory arises from the invalidity of neglecting transverse shear deformation in CPT. Transverse shear is no longer negligible in thick plates of length to thickness ratios less than 10 for isotropic plates. Also, shear deformation is significant in composites with length to thickness ratios much larger than 10. This is due to the effective elastic modulus along the fiber direction (E_1) being much larger than the transverse shear moduli (G_{13} , G_{23}), sometimes by the order of 25 to 40 compared to 2.6 for a representative isotropic material [22].

The assumptions listed in the next section describe the restrictions to this model and should be considered when using this program for engineering applications. The global axes described in the next section are shown in Figure 8 with

an example rectangular plate. Note that the xy-plane corresponds to the laminate midplane and the z-coordinate describes the distance and direction (upward or downward) of a point with respect to this plane.

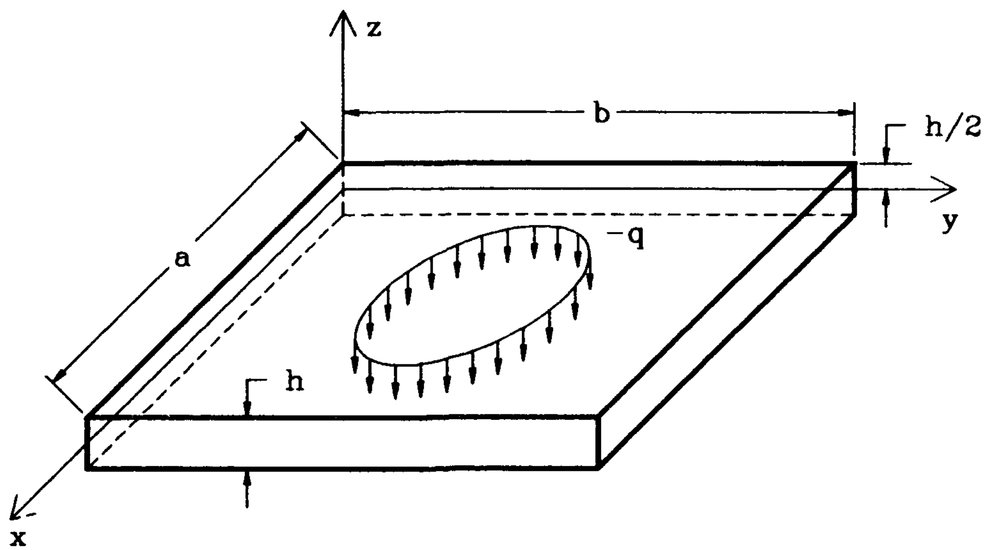

Figure 8: Schematic of a rectangular plate.

Assumptions

The basic assumptions for this first-order shear deformation theory (FSDT) are given as follows. The terms used in the assumptions are described on the following pages [33].

1. The plate is constructed of an arbitrary number of orthotropic layers (laminae) which are perfectly boned together. The directions of principle orthotropic material symmetry, the thickness, and the material of each layer may vary.
2. The plate is considered to be relatively thin compared to face dimensions.
3. Plate displacements (u, v, w) are small compared to the plate thickness (h).
4. In-plane strains (ε_x, ε_y, γ_{xy}) are small.

5. To include in-plane force effects, non-linear terms in the equations of motion involving products of stresses and plate slopes are retained and all other non-linear terms are neglected.
6. Transverse shear strain (γ_{yz}, γ_{xz}) are included in the analysis in case they are not negligible.
7. The total in-plane displacements (u_1, u_2) are linear functions of the z-coordinate through the plate thickness.
8. Transverse normal strain (ε_z) is negligible compared to other strains.
9. Each lamina behaves in an elastic manner and is governed by Hooke's law.
10. The total plate thickness is uniform throughout the plate.
11. Body forces are negligible compared to other plate forces.
12. All linear inertial terms are retained for dynamic analysis.

Variational Energy Formulation

For FSDT, a first-order (linear) displacement field in terms of z is assumed. The general displacement of any point in the plate is described by the following first-order displacement functions.

$$\begin{aligned} u_1(x,y,z,t) &= u(x,y,t) + z\psi_x(x,y,t) \\ u_2(x,y,z,t) &= v(x,y,t) + z\psi_y(x,y,t) \\ u_3(x,y,z,t) &= w(x,y,t) \end{aligned} \tag{3-1}$$

where u_1, u_2, and u_3 are the displacements in the x, y, and z directions respectively, u, v, and w are the displacements of the laminate mid-plane in the same directions, and Ψ_x and Ψ_y are the rotations in the xz and yz planes respectively caused by plate bending and transverse shear deformation. By adhering to the assumption of small

displacements the strains are derived from the displacement functions in equation 3-1 as follows:

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \psi_x}{\partial x} = \varepsilon_x^\circ + z\kappa_x \\
\varepsilon_y &= \frac{\partial u_2}{\partial y} = \frac{\partial v}{\partial y} + z\frac{\partial \psi_y}{\partial y} = \varepsilon_y^\circ + z\kappa_y \\
\gamma_{xy} &= \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) = \gamma_{xy}^\circ + z\kappa_{xy} \\
\gamma_{yz} &= \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \psi_y + \frac{\partial w}{\partial y} = \gamma_{yz}^{avg} \\
\gamma_{xz} &= \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \psi_x + \frac{\partial w}{\partial x} = \gamma_{xz}^{avg} \\
\varepsilon_z &= \frac{\partial u_3}{\partial z} = 0
\end{aligned}
\tag{3-2}
$$

The definitions of the mid-plane strains $(\varepsilon_x^\circ, \varepsilon_y^\circ, \gamma_{xy}^\circ)$ and the curvatures $(\kappa_x, \kappa_y, \kappa_{xy})$ can easily be derived from the last two equalities in each equation. Note that the transverse shear strains $(\gamma_{yz}, \gamma_{xz})$ are constant through the plate thickness (independent of z). Since actual transverse shear strains are not constant through the plate thickness, the ones predicted by this theory represent average shear strains.

This FSDT is based on a displacement derivation. For each point in the laminate mid-plane, five degrees of freedom are defined. This allows for enough degrees of freedom to provide adequate results for a majority of composite plate applications. These degrees of freedom are also referred to as generalized

displacements and consist of the following displacements and rotations: u, v, w, ψ_x and ψ_y. Each generalized displacement is a function only of mid-plane position (x, y) and time (t) for dynamic cases. These generalized displacements allow for the reduction of three-dimensional model (x,y,z) to a two-dimensional model (x,y) of the laminate mid-plane for analysis purposes.

For the general dynamic case, the energy formulation is based on Hamilton's Principle:

$$\int_{t_0}^{t_1} \delta L \, dt = 0 \tag{3-3}$$

The Lagrangian (L) may be defined as components of energy in the following manner.

$$L = U + V - T \tag{3-4}$$

where U is the total strain energy, V is the potential energy due to the uniformly distributed transverse load, and T is the total kinetic energy of the plate.

The first variation of the Lagrangian can be written as the variation of its components:

$$\delta L = \delta U + \delta V - \delta T \tag{3-5}$$

In order to find the first variation of the Lagrangian, the first variation of each of the components are derived in terms of displacements. The total strain energy of the plate (U) is defined by the integration of the strain energy in terms of stresses and strains of each point in the plate.

$$U=\tfrac{1}{2}\iiint_V\left(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{xz}\gamma_{xz}\right)dV \tag{3-6}$$

Using the definitions of stress from equation 2-3 in the previous chapter, the strain energy can be recast as:

$$\begin{aligned}U=\tfrac{1}{2}\iiint_V\Big[&\overline{Q}_{11}\varepsilon_x^2+\overline{Q}_{22}\varepsilon_y^2+\overline{Q}_{33}\gamma_{xy}^2+2\overline{Q}_{12}\varepsilon_x\varepsilon_y+2\overline{Q}_{13}\varepsilon_x\gamma_{xy}+2\overline{Q}_{12}\varepsilon_x\varepsilon_y\\&+2\overline{Q}_{23}\varepsilon_y\gamma_{xy}+\overline{Q}_{44}\gamma_{yz}^2+\overline{Q}_{55}\gamma_{xz}^2+2\overline{Q}_{45}\gamma_{yz}\gamma_{xz}\Big]dV\end{aligned} \tag{3-7}$$

Substituting the definitions of strain from equation 3-2, the following equation is obtained:

$$\begin{aligned}U=\frac{1}{2}\iiint_V\Bigg[&\overline{Q}_{11}\left(\frac{\partial u}{\partial x}+z\frac{\partial\psi_x}{\partial x}\right)^2+\overline{Q}_{22}\left(\frac{\partial v}{\partial y}+z\frac{\partial\psi_y}{\partial y}\right)^2+\overline{Q}_{33}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+z\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\right)^2\\&+2\overline{Q}_{12}\left(\frac{\partial u}{\partial x}+z\frac{\partial\psi_x}{\partial x}\right)\left(\frac{\partial v}{\partial y}+z\frac{\partial\psi_y}{\partial y}\right)+2\overline{Q}_{13}\left(\frac{\partial u}{\partial x}+z\frac{\partial\psi_x}{\partial x}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+z\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\right)\\&+2\overline{Q}_{23}\left(\frac{\partial v}{\partial y}+z\frac{\partial\psi_y}{\partial y}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+z\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\right)+\overline{Q}_{44}\left(\psi_y+\frac{\partial w}{\partial y}\right)^2\\&+\overline{Q}_{55}\left(\psi_x+\frac{\partial w}{\partial x}\right)^2+2\overline{Q}_{45}\left(\psi_x+\frac{\partial w}{\partial x}\right)\left(\psi_y+\frac{\partial w}{\partial y}\right)\Bigg]dV\end{aligned} \tag{3-8}$$

Expanding the equation and factoring terms of z $(1, z, z^2)$ yields the following:

$$U=\frac{1}{2}\iiint_V\Bigg[\overline{Q}_{11}\left(\frac{\partial u}{\partial x}^2+2z\frac{\partial\psi_x}{\partial x}+z^2\frac{\partial\psi_x}{\partial x}^2\right)+\overline{Q}_{22}\left(\frac{\partial v}{\partial y}^2+2z\frac{\partial\psi_y}{\partial y}+z^2\frac{\partial\psi_y}{\partial y}^2\right)+\overline{Q}_{33}$$

$$\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+z\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\right)+2\overline{Q}_{12}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+z\left(\frac{\partial u}{\partial x}\frac{\partial\psi_y}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial\psi_x}{\partial x}\right)+z^2\frac{\partial\psi_x}{\partial x}\frac{\partial\psi_y}{\partial y}\right)$$

$$+2\overline{Q}_{13}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+z\left(\frac{\partial u}{\partial x}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)+\frac{\partial\psi_x}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)\right.$$

$$\left.+z^2\frac{\partial\psi_x}{\partial x}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\right)+2\overline{Q}_{23}\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial y}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\right.$$

$$\left.+z\frac{\partial\psi_y}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+z^2\frac{\partial\psi_y}{\partial y}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\right)+\overline{Q}_{44}\left(\psi_y+\frac{\partial w}{\partial y}\right)^2+\overline{Q}_{55}\left(\psi_x+\frac{\partial w}{\partial x}\right)^2$$

$$+2\overline{Q}_{45}\left(\psi_x+\frac{\partial w}{\partial x}\right)\left(\psi_y+\frac{\partial w}{\partial y}\right)\Bigg]dV \tag{3-9}$$

Since u, v, w, ψ_x and ψ_y are independent of z, the triple integral may be integrated with respect to z, and the relations for A_{ij}, B_{ij}, and D_{ij} from equations 2-10 and 2-11 may be applied to yield:

$$U=\frac{1}{2}\iint_A\Bigg[A_{11}\frac{\partial u}{\partial x}^2+2B_{11}\frac{\partial u}{\partial x}\frac{\partial\psi_x}{\partial x}+D_{11}\frac{\partial\psi_x}{\partial x}^2+A_{22}\frac{\partial v}{\partial y}^2+2B_{22}\frac{\partial v}{\partial y}\frac{\partial\psi_y}{\partial y}+D_{22}\frac{\partial\psi_y}{\partial y}^2$$

$$+A_{33}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+2B_{33}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)+D_{33}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)^2$$

$$+2A_{12}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+B_{12}\left(\frac{\partial u}{\partial x}\frac{\partial\psi_y}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial\psi_x}{\partial x}\right)+D_{12}\frac{\partial\psi_x}{\partial x}\frac{\partial\psi_y}{\partial y}+2A_{13}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right)$$

$$+2B_{13}\left(\frac{\partial u}{\partial x}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)+\frac{\partial\psi_x}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+2D_{13}\frac{\partial\psi_x}{\partial x}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)$$

$$+2A_{23}\frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+2B_{23}\left(\frac{\partial v}{\partial y}\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right)+\frac{\partial \psi_y}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)$$

$$+2D_{23}\frac{\partial \psi_y}{\partial y}\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right)+A_{44}\left(\psi_y+\frac{\partial w}{\partial y}\right)^2+A_{55}\left(\psi_x+\frac{\partial w}{\partial x}\right)^2$$

$$+2A_{45}\left(\psi_x+\frac{\partial w}{\partial x}\right)\left(\psi_y+\frac{\partial w}{\partial y}\right)\Bigg]dA \tag{3-10}$$

The first variation of the strain energy is:

$$\delta U=\iint_A\Bigg\{\Bigg[A_{11}\frac{\partial u}{\partial x}+A_{12}\frac{\partial v}{\partial y}+A_{13}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+B_{11}\frac{\partial \psi_x}{\partial x}+B_{12}\frac{\partial \psi_y}{\partial y}$$

$$+B_{13}\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right)\Bigg]\frac{\partial}{\partial x}(\delta u)+\Bigg[A_{12}\frac{\partial u}{\partial x}+A_{22}\frac{\partial v}{\partial y}+A_{23}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$+B_{12}\frac{\partial \psi_x}{\partial x}+B_{22}\frac{\partial \psi_y}{\partial y}+B_{23}\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right)\Bigg]\frac{\partial}{\partial y}(\delta u)+\Bigg[A_{13}\frac{\partial u}{\partial x}+A_{23}\frac{\partial v}{\partial y}$$

$$+A_{33}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+B_{13}\frac{\partial \psi_x}{\partial x}+B_{23}\frac{\partial \psi_y}{\partial y}+B_{33}\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right)\Bigg]\left(\frac{\partial}{\partial y}(\delta u)+\frac{\partial}{\partial x}(\delta v)\right)$$

$$+\Bigg[B_{11}\frac{\partial u}{\partial x}+B_{12}\frac{\partial v}{\partial y}+B_{13}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+D_{11}\frac{\partial \psi_x}{\partial x}+D_{12}\frac{\partial \psi_y}{\partial y}+D_{13}\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right)\Bigg]$$

$$\frac{\partial}{\partial x}(\delta\psi_x)+\Bigg[B_{12}\frac{\partial u}{\partial x}+B_{22}\frac{\partial v}{\partial y}+B_{23}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+D_{12}\frac{\partial \psi_x}{\partial x}+D_{22}\frac{\partial \psi_y}{\partial y}$$

$$+D_{13}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\Bigg]\frac{\partial}{\partial y}(\delta\psi_y)+\Bigg[B_{13}\frac{\partial u}{\partial x}+B_{23}\frac{\partial v}{\partial y}+B_{33}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+D_{13}\frac{\partial\psi_x}{\partial x}$$

$$+D_{23}\frac{\partial\psi_y}{\partial y}+D_{33}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)\Bigg]\left(\frac{\partial}{\partial y}(\delta\psi_x)+\frac{\partial}{\partial x}(\delta\psi_y)\right)+\Bigg[A_{55}\left(\psi_x+\frac{\partial w}{\partial x}\right)$$

$$+A_{45}\left(\psi_y+\frac{\partial w}{\partial y}\right)\Bigg]\frac{\partial}{\partial x}(\delta w)+\Bigg[A_{45}\left(\psi_x+\frac{\partial w}{\partial x}\right)+A_{45}\left(\psi_y+\frac{\partial w}{\partial y}\right)\Bigg]\frac{\partial}{\partial y}(\delta w)\Bigg\}dA$$

(3-11)

Using the expression for the derivative of a product, the following relation is derived:

$$\iint_A\left(f_1\frac{\partial}{\partial x}(\delta u)+f_2\frac{\partial}{\partial y}(\delta u)\right)dA=-\iint_A\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)\delta u\,dA+\iint_A\left(\frac{\partial}{\partial x}(f_1\delta u)+\frac{\partial}{\partial y}(f_2\delta u)\right)dA$$

(3-12)

Green's Theorem is used to evaluate the last integral above:

$$\iint_A\left(\frac{\partial}{\partial x}(f_1\delta u)+\frac{\partial}{\partial y}(f_2\delta u)\right)dA=\int_S\left(f_1\delta u\,dy-f_2\delta u\,dx\right) \qquad (3\text{-}13)$$

This leads to the following equation used to evaluate δU:

$$\iint_A\left(f_1\frac{\partial}{\partial x}(\delta u)+f_2\frac{\partial}{\partial y}(\delta u)\right)dA=-\iint_A\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)\delta u\,dA+\int_S\left(f_1\delta u\,dy-f_2\delta u\,dx\right) \qquad (3\text{-}14)$$

δU may be divided into terms with common factors of variational displacement, $\delta U_{\delta u}$ to $\delta U_{\delta \psi_y}$, and a boundary term, δU_S.

$$\delta U = \delta U_{\delta u} + \delta U_{\delta v} + \delta U_{\delta w} + \delta U_{\delta \psi_x} + \delta U_{\delta \psi_y} + \delta U_S \tag{3-15}$$

Thus,

$$\begin{aligned} \delta U_{\delta u} = -\iint_A \Bigg[& A_{11}\frac{\partial^2 u}{\partial x^2} + A_{12}\frac{\partial^2 v}{\partial x \partial y} + A_{13}\left(2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + A_{23}\frac{\partial^2 v}{\partial y^2} \\ & + A_{33}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + B_{11}\frac{\partial^2 \psi_x}{\partial x^2} + B_{12}\frac{\partial^2 \psi_y}{\partial x \partial y} + B_{13}\left(2\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2}\right) \\ & + B_{23}\frac{\partial^2 \psi_y}{\partial y^2} + B_{33}\left(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y}\right)\Bigg] \delta u \, dA \end{aligned} \tag{3-16}$$

$$\begin{aligned} \delta U_{\delta v} = -\iint_A \Bigg[& A_{12}\frac{\partial^2 u}{\partial x \partial y} + A_{22}\frac{\partial^2 v}{\partial x \partial y} + A_{23}\left(\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x \partial y}\right) + A_{13}\frac{\partial^2 u}{\partial x^2} \\ & + A_{33}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + B_{12}\frac{\partial^2 \psi_x}{\partial x \partial y} + B_{13}\frac{\partial^2 \psi_x}{\partial x^2} + B_{22}\frac{\partial^2 \psi_y}{\partial y^2} \\ & + B_{23}\left(\frac{\partial^2 \psi_x}{\partial y^2} + 2\frac{\partial^2 \psi_y}{\partial x \partial y}\right) + B_{33}\left(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2}\right)\Bigg] \delta v \, dA \end{aligned} \tag{3-17}$$

$$\delta U_{\delta w} = -\iint_A \left[A_{44}\left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) + A_{55}\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \right.$$
$$\left. + A_{45}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y} \right) \right] \delta w \, dA \qquad (3\text{-}18)$$

$$\delta U_{\delta \psi_x} = -\iint_A \left[B_{11}\frac{\partial^2 u}{\partial x^2} + B_{12}\frac{\partial^2 v}{\partial x \partial y} + B_{13}\left(2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} \right) + B_{23}\frac{\partial^2 v}{\partial y^2} \right.$$
$$+ B_{33}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + D_{11}\frac{\partial^2 \psi_x}{\partial x^2} + D_{12}\frac{\partial^2 \psi_y}{\partial x \partial y} + D_{13}\left(2\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2} \right) \qquad (3\text{-}19)$$
$$\left. + D_{23}\frac{\partial^2 \psi_y}{\partial y^2} + D_{33}\left(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y} \right) \right] \delta \psi_x \, dA$$

and

$$\delta U_{\delta \psi_y} = -\iint_A \left[B_{12}\frac{\partial^2 u}{\partial x \partial y} + B_{22}\frac{\partial^2 v}{\partial x \partial y} + B_{23}\left(\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x \partial y} \right) + B_{13}\frac{\partial^2 u}{\partial x^2} \right.$$
$$+ B_{33}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} \right) + D_{12}\frac{\partial^2 \psi_x}{\partial x \partial y} + D_{13}\frac{\partial^2 \psi_x}{\partial x^2} + D_{22}\frac{\partial^2 \psi_y}{\partial y^2} \quad . \qquad (3\text{-}20)$$
$$\left. + D_{23}\left(\frac{\partial^2 \psi_x}{\partial y^2} + 2\frac{\partial^2 \psi_y}{\partial x \partial y} \right) + D_{33}\left(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2} \right) \right] \delta \psi_y \, dA$$

By utilizing Green's Theorem and applying the laminate load resultant relations from equations 2-8 and 2-9, the boundary condition component becomes:

$$\delta U_S = \int_{S_X}\left(N_x\delta u + N_{xy}\delta v + Q_x\delta w + M_x\delta\psi_x + M_{xy}\delta\psi_y\right)dy$$
$$-\int_{S_Y}\left(N_{xy}\delta u + N_y\delta v + Q_y\delta w + M_{xy}\delta\psi_x + M_y\delta\psi_y\right)dx \tag{3-21}$$

For the computer program, the loading on the plate is assumed to be a uniform pressure, thus the potential energy in the plate due to applied transverse pressure (V) is:

$$V = \iint_A q\, w\, dA \tag{3-22}$$

and the first variation of V is

$$\delta V = \iint_A q\, \delta w\, dA \tag{3-23}$$

The total kinetic energy of the plate (T) is the final component of the Lagrangian. It consists of the following integral of the energy of each point in the plate.

$$T = \tfrac{1}{2}\iiint_V \rho\left[\frac{\partial u_1}{\partial t}^2 + \frac{\partial u_2}{\partial t}^2 + \frac{\partial u_3}{\partial t}^2\right]dV \tag{3-24}$$

The time derivatives of the displacements are found by taking the first time derivative of equation 3-1 and result as follows:

$$\frac{\partial u_1}{\partial t} = \frac{\partial u}{\partial t} + z\frac{\partial \psi_x}{\partial t}$$

$$\frac{\partial u_1}{\partial t} = \frac{\partial v}{\partial t} + z\frac{\partial \psi_y}{\partial t} \tag{3-25}$$

$$\frac{\partial u_3}{\partial t} = \frac{\partial w}{\partial t}$$

Substitution of these time derivatives into equation 3-24 yields:

$$T = \tfrac{1}{2}\iiint_V \rho\left[\frac{\partial u}{\partial t}^2 + \frac{\partial v}{\partial t}^2 + \frac{\partial w}{\partial t}^2 + 2z\left(\frac{\partial u}{\partial t}\frac{\partial \psi_x}{\partial t} + \frac{\partial v}{\partial t}\frac{\partial \psi_y}{\partial t}\right) + z^2\left(\frac{\partial \psi_x}{\partial t}^2 + \frac{\partial \psi_y}{\partial t}^2\right)\right]dV \tag{3-26}$$

Since the time derivatives of u, v, w, Ψ_x and Ψ_y are independent of z, the equation may be integrated with respect to z and the inertial terms from equation 2-12 (ρ_1, ρ_2, and ρ_3) may be applied:

$$T = \tfrac{1}{2}\iint_A\left[\rho_1\left(\frac{\partial u}{\partial t}^2 + \frac{\partial v}{\partial t}^2 + \frac{\partial w}{\partial t}^2\right) + 2\rho_2\left(\frac{\partial u}{\partial t}\frac{\partial \psi_x}{\partial t} + \frac{\partial v}{\partial t}\frac{\partial \psi_y}{\partial t}\right) + \rho_3\left(\frac{\partial \psi_x}{\partial t}^2 + \frac{\partial \psi_y}{\partial t}^2\right)\right]dA \tag{3-27}$$

Taking the first variation of the kinetic energy yields:

$$\begin{aligned}\delta T = \iint_A\Bigg[&\left(\rho_1\frac{\partial u}{\partial t} + \rho_2\frac{\partial \psi_x}{\partial t}\right)\frac{\partial}{\partial t}(\delta u) + \left(\rho_1\frac{\partial v}{\partial t} + \rho_2\frac{\partial \psi_y}{\partial t}\right)\frac{\partial}{\partial t}(\delta u) + \rho_1\frac{\partial w}{\partial t}\frac{\partial}{\partial t}(\delta w)\\ &+\left(\rho_2\frac{\partial u}{\partial t} + \rho_3\frac{\partial \psi_x}{\partial t}\right)\frac{\partial}{\partial t}(\delta \psi_x) + \left(\rho_2\frac{\partial v}{\partial t} + \rho_3\frac{\partial \psi_y}{\partial t}\right)\frac{\partial}{\partial t}(\delta \psi_y)\Bigg]dA\end{aligned} \tag{3-28}$$

and using the same relation as the one derived for strain energy (equation 3-12), δT becomes:

$$\delta T = -\iint_A \left[\left(\rho_1 \frac{\partial^2 u}{\partial t^2} + \rho_2 \frac{\partial^2 \psi_x}{\partial t^2} \right) \delta u + \left(\rho_1 \frac{\partial^2 v}{\partial t^2} + \rho_2 \frac{\partial^2 \psi_y}{\partial t^2} \right) \delta v + \rho_1 \frac{\partial^2 w}{\partial t^2} \delta w \right.$$
$$\left. + \left(\rho_2 \frac{\partial^2 u}{\partial t^2} + \rho_3 \frac{\partial^2 \psi_x}{\partial t^2} \right) \delta \psi_x + \left(\rho_2 \frac{\partial^2 v}{\partial t^2} + \rho_3 \frac{\partial^2 \psi_y}{\partial t^2} \right) \delta \psi_y \right] dA$$

$$+ \iint_A \frac{\partial}{\partial t} \left[\left(\rho_1 \frac{\partial u}{\partial t} + \rho_2 \frac{\partial \psi_x}{\partial t} \right) \delta u + \left(\rho_1 \frac{\partial v}{\partial t} + \rho_2 \frac{\partial \psi_y}{\partial t} \right) \delta v + \rho_1 \frac{\partial w}{\partial t} \delta w \right.$$
$$\left. + \left(\rho_2 \frac{\partial u}{\partial t} + \rho_3 \frac{\partial \psi_x}{\partial t} \right) \delta \psi_x + \left(\rho_2 \frac{\partial v}{\partial t} + \rho_3 \frac{\partial \psi_y}{\partial t} \right) \delta \psi_y \right] dA \tag{3-29}$$

The above variational components of virtual energy (δU, δV, δT) may be combined to form the Lagrangian and collected on terms with common variational displacement factors.

$$\int_{t_0}^{t_1} \delta L dt + \delta L_t \Big|_{t_0}^{t_1} = 0 \tag{3-30}$$

where $\delta L = \delta L_{\delta u} + \delta L_{\delta v} + \delta L_{\delta w} + \delta L_{\delta \psi_x} + \delta L_{\delta \psi_y} + \delta L_s$ and δL_t is defined at the time limits t_o and t_1.

Therefore,

$$\delta L_{\delta u} = -\iint_A \left[A_{11}\frac{\partial^2 u}{\partial x^2} + A_{12}\frac{\partial^2 v}{\partial x \partial y} + A_{13}\left(2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + A_{23}\frac{\partial^2 v}{\partial y^2} \right.$$
$$+ A_{33}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + B_{11}\frac{\partial^2 \psi_x}{\partial x^2} + B_{12}\frac{\partial^2 \psi_y}{\partial x \partial y} + B_{13}\left(2\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2}\right) \tag{3-31}$$
$$\left. + B_{23}\frac{\partial^2 \psi_y}{\partial y^2} + B_{33}\left(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y}\right) - \rho_1\frac{\partial^2 u}{\partial t^2} - \rho_2\frac{\partial^2 \psi_x}{\partial t^2} \right] \delta u \, dA$$

$$\delta L_{\delta v} = -\iint_A \left[A_{12}\frac{\partial^2 u}{\partial x \partial y} + A_{22}\frac{\partial^2 v}{\partial x \partial y} + A_{23}\left(\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x \partial y}\right) + A_{13}\frac{\partial^2 u}{\partial x^2} \right.$$
$$+ A_{33}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + B_{12}\frac{\partial^2 \psi_x}{\partial x \partial y} + B_{13}\frac{\partial^2 \psi_x}{\partial x^2} + B_{22}\frac{\partial^2 \psi_y}{\partial y^2} + B_{23}\left(\frac{\partial^2 \psi_x}{\partial y^2} + 2\frac{\partial^2 \psi_y}{\partial x \partial y}\right)$$
$$\left. + B_{33}\left(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2}\right) - \rho_1\frac{\partial^2 v}{\partial t^2} - \rho_2\frac{\partial^2 \psi_y}{\partial t^2} \right] \delta v \, dA \tag{3-32}$$

$$\delta L_{\delta w} = -\iint_A \left[A_{44}\left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + A_{55}\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + A_{45}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right) \right.$$
$$\left. + A_{45}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right) - \rho_1\frac{\partial^2 w}{\partial t^2} - q \right] \delta w \, dA \tag{3-33}$$

$$
\begin{aligned}
\delta L_{\delta\psi_x} = -\iint_A \Bigg[& B_{11}\frac{\partial^2 u}{\partial x^2} + B_{12}\frac{\partial^2 v}{\partial x \partial y} + B_{13}\left(2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + B_{23}\frac{\partial^2 v}{\partial y^2} \\
& + B_{33}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + D_{11}\frac{\partial^2 \psi_x}{\partial x^2} + D_{12}\frac{\partial^2 \psi_y}{\partial x \partial y} + D_{13}\left(2\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2}\right) \\
& + D_{23}\frac{\partial^2 \psi_y}{\partial y^2} + D_{33}\left(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y}\right) - \rho_2\frac{\partial^2 u}{\partial t^2} - \rho_3\frac{\partial^2 \psi_x}{\partial t^2}\Bigg]\delta\psi_x \, dA
\end{aligned} \tag{3-34}
$$

and

$$
\begin{aligned}
\delta L_{\delta\psi_y} = -\iint_A \Bigg[& B_{12}\frac{\partial^2 u}{\partial x \partial y} + B_{22}\frac{\partial^2 v}{\partial x \partial y} + B_{23}\left(\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x \partial y}\right) + B_{13}\frac{\partial^2 u}{\partial x^2} \\
& + B_{33}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + D_{12}\frac{\partial^2 \psi_x}{\partial x \partial y} + D_{13}\frac{\partial^2 \psi_x}{\partial x^2} + D_{22}\frac{\partial^2 \psi_y}{\partial y^2} \\
& + D_{23}\left(\frac{\partial^2 \psi_x}{\partial y^2} + 2\frac{\partial^2 \psi_y}{\partial x \partial y}\right) + D_{33}\left(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2}\right) - \rho_2\frac{\partial^2 v}{\partial t^2} - \rho_3\frac{\partial^2 \psi_y}{\partial t^2}\Bigg]\delta\psi_y \, dA
\end{aligned} \tag{3-35}
$$

The boundary term of the Lagrangian can be recast in the following general form along the boundary.

$$
\delta L_S = \int_S \left(N_n \delta u_n + N_{ns} \delta u_s + Q_n \delta w + M_n \delta\psi_n + M_{ns} \delta\psi_s\right) dS \tag{3-36}
$$

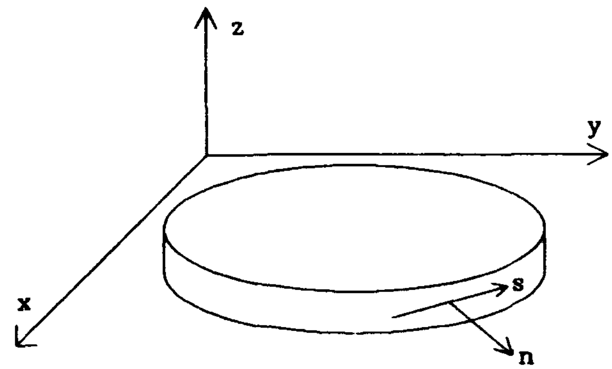

Figure 9: Orientation of n and s axes along the plate boundary.

In order to satisfy Hamilton's principle, one of the products of each term must be specified along the boundary over the entire time interval. This leads to the following conditions which must be specified on the plate boundaries. The orientations of n and s along the boundary are shown in Figure 9.

$$N_n \text{ or } u_n,\quad N_{ns} \text{ or } u_s,\quad Q_n \text{ or } w,\quad M_n \text{ or } \psi_n,\quad M_{ns} \text{ or } \psi_s \tag{3-37}$$

The final term of the Lagrangian variation (δL_t) is defined only at the limits of the time integral. In order for this term to satisfy Hamilton's principle, the generalized displacements must be specified at the time interval endpoints. This condition is satisfied in the finite element method by discretizing the time interval and treating each time step in a semi-static sense as explained in chapter IV. The generalized displacements are then found through a static analysis at the time limits. This term is shown below as:

$$\delta L_t = \iint_A \left[\left(\rho_1 \frac{\partial u}{\partial t} + \rho_2 \frac{\partial \psi_x}{\partial t} \right) \delta u + \left(\rho_1 \frac{\partial v}{\partial t} + \rho_2 \frac{\partial \psi_y}{\partial t} \right) \delta v + \rho_1 \frac{\partial w}{\partial t} \delta w \right.$$
$$\left. + \left(\rho_2 \frac{\partial u}{\partial t} + \rho_3 \frac{\partial \psi_x}{\partial t} \right) \delta \psi_x + \left(\rho_2 \frac{\partial v}{\partial t} + \rho_3 \frac{\partial \psi_y}{\partial t} \right) \delta \psi_y \right] dA \qquad (3\text{-}38)$$

The combination of all these variational terms yields the equation form given in equation 3-39. The first five terms in the area integral (first line below) each include the integral of a product of a term in parantheses and a generalized variational displacement. In order to satisfy Hamilton's principle, each of the terms in parantheses must be equal to zero since the variational displacements are arbitrary. This process generates five governing equations of motion. The second line in equation 3-39 defines the boundary and initial conditions.

$$\int_{t_0}^{t_1} \left\{ \iint_A \left[(\quad) \delta u + (\quad) \delta v + (\quad) \delta w + (\quad) \delta \psi_x + (\quad) \delta \psi_y \right] dA \right.$$
$$\left. + \int_S \left[N_n \delta u_n + N_s \delta u_s + Q_n \delta w + M_n \delta \psi_n + M_s \delta \psi_s \right] dS \right\} + \delta L_t \Big|_{t_0}^{t_1} = 0 \qquad (3\text{-}39)$$

By applying the definition of strain in equation 3-2 and the load resultant definitions from equations 2-8 and 2-9, the equations of motion can be displayed, in a shorter notation, in terms of load resultants. The variational displacement before the equations below describe the paratheses location of the corresponding equation in equation 3-39.

$$\delta u:\quad \frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=\rho_1\frac{\partial^2 u}{\partial t^2}+\rho_2\frac{\partial^2 \psi_x}{\partial t^2}$$

$$\delta v:\quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=\rho_1\frac{\partial^2 v}{\partial t^2}+\rho_2\frac{\partial^2 \psi_y}{\partial t^2}$$

$$\delta w:\quad \frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=\rho_1\frac{\partial^2 w}{\partial t^2}+q \qquad (3\text{-}40)$$

$$\delta \psi_x:\quad \frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}=\rho_2\frac{\partial^2 u}{\partial t^2}+\rho_3\frac{\partial^2 \psi_x}{\partial t^2}$$

$$\delta \psi_y:\quad \frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=\rho_2\frac{\partial^2 v}{\partial t^2}+\rho_3\frac{\partial^2 \psi_y}{\partial t^2}$$

These five equations are the governing equations for all plates based on small deflection theory including shear deformation and rotary inertia. For the finite element method presented in the next chapter, a displacement formulation is required. The equations of motion can be cast in matrix form in terms of displacements for easier conversion to the finite element method. This leads to an equation of the following form:

$$[M]\{\ddot{\Delta}\}+[K]\{\Delta\}=\{f\} \qquad (3\text{-}41)$$

where:

$$\{\Delta\}=\{u,v,w,\psi_x,\psi_y\}^T$$
$$\{\ddot{\Delta}\}=\{\ddot{u},\ddot{v},\ddot{w},\ddot{\psi}_x,\ddot{\psi}_y\}^T \qquad (3\text{-}42)$$
$$\{f\}=\{0,0,q,0,0\}^T$$

The mass matrix terms are shown below:

$$
\begin{aligned}
&M_{11} = M_{22} = M_{33} = \rho_1 \\
&M_{14} = M_{25} = M_{41} = M_{52} = \rho_2 \\
&M_{44} = M_{55} = \rho_3 \\
&\text{All other } M_{ij} = 0
\end{aligned}
\qquad (3\text{-}43)
$$

The stiffness matrix terms are written as <u>differential operators</u> on the vector $\{\Delta\}$. Note that this matrix is symmetric and only the upper half terms are indicated below.

$$
\begin{aligned}
K_{11} &= A_{11}\tfrac{\partial^2}{\partial x^2} + 2A_{13}\tfrac{\partial^2}{\partial x\partial y} + A_{33}\tfrac{\partial^2}{\partial y^2} \\
K_{12} &= A_{13}\tfrac{\partial^2}{\partial x^2} + (A_{12} + A_{33})\tfrac{\partial^2}{\partial x\partial y} + A_{23}\tfrac{\partial^2}{\partial y^2} \\
K_{13} &= 0 \\
K_{14} &= B_{11}\tfrac{\partial^2}{\partial x^2} + 2B_{13}\tfrac{\partial^2}{\partial x\partial y} + B_{33}\tfrac{\partial^2}{\partial y^2} \\
K_{15} &= B_{13}\tfrac{\partial^2}{\partial x^2}(B_{12} + B_{33})\tfrac{\partial^2}{\partial x\partial y} + B_{23}\tfrac{\partial^2}{\partial y^2} \\
K_{22} &= A_{33}\tfrac{\partial^2}{\partial x^2} + A_{23}\tfrac{\partial^2}{\partial x\partial y} + A_{22}\tfrac{\partial^2}{\partial y^2} \\
K_{23} &= 0 \\
K_{24} &= B_{13}\tfrac{\partial^2}{\partial x^2} + (B_{12} + B_{33})\tfrac{\partial^2}{\partial x\partial y} + B_{23}\tfrac{\partial^2}{\partial y^2} \\
K_{25} &= B_{33}\tfrac{\partial^2}{\partial x^2} + 2B_{23}\tfrac{\partial^2}{\partial x\partial y} + B_{22}\tfrac{\partial^2}{\partial y^2} \\
K_{33} &= A_{55}\tfrac{\partial^2}{\partial x^2} + 2A_{45}\tfrac{\partial^2}{\partial x\partial y} + A_{44}\tfrac{\partial^2}{\partial y^2} \\
K_{34} &= A_{55}\tfrac{\partial}{\partial x} + A_{45}\tfrac{\partial}{\partial y} \\
K_{35} &= A_{45}\tfrac{\partial}{\partial x} + A_{44}\tfrac{\partial}{\partial y} \\
K_{44} &= D_{11}\tfrac{\partial^2}{\partial x^2} + 2D_{13}\tfrac{\partial^2}{\partial x\partial y} + D_{33}\tfrac{\partial^2}{\partial y^2} + A_{55} \\
K_{45} &= D_{13}\tfrac{\partial^2}{\partial x^2} + (D_{12} + D_{33})\tfrac{\partial^2}{\partial x\partial y} + D_{23}\tfrac{\partial^2}{\partial y^2} + A_{45} \\
K_{55} &= D_{33}\tfrac{\partial^2}{\partial x^2} + 2D_{23}\tfrac{\partial^2}{\partial x\partial y} + D_{22}\tfrac{\partial^2}{\partial y^2} + A_{44}
\end{aligned}
\qquad (3\text{-}44)
$$

With the FSDT presented in this chapter, the ground work is set to implement this theory into finite element code. The next chapter shows how the above matrix equation is discretized into finite elements and applied to the computer program.

CHAPTER IV

FINITE ELEMENT FORMULATION

This chapter describes the finite element method (FEM) formulation used in the computer program. Most of the FSDT development is presented in the last chapter. The subsequent FEM formulation follows J. N. Reddy [25]. The details shown below describe how the previous governing equations are discretized into elements. It is assumed that the reader has some knowledge of the finite element method including interpolation functions.

Generalized Displacements and Interpolation Functions

For the finite element model, the domain of the plate mid-plane is denoted as R and is divided into a finite number of elements whose domains are R_e (where e = 1,2,3,... number of elements). For each element domain, R_e , the generalized displacements are defined by use of an interpolation function ϕ_i (where i = 1, 2, 3, ..., n). The interpolation function is the same for all five generalized displacements which are described by:

$$u = \sum_{i=1}^{n} u_i \phi_i \qquad v = \sum_{i=1}^{n} v_i \phi_i \qquad w = \sum_{i=1}^{n} w_i \phi_i$$
$$\psi_x = \sum_{i=1}^{n} \psi_{xi} \phi_i \qquad \psi_y = \sum_{i=1}^{n} \psi_{yi} \phi_i \tag{4-1}$$

n is the number of nodes per element and the interpolation function depends on the type of element used in the analysis.

Development of Element Mass and Stiffness Matrices

The equations of motion derived for the FSDT can be applied to each element by substituting the equations 4-1 and into equations 3-41 to 3-44. The simplified form of the differential equation 4-2 remains the same, but the size and formulation of the matrices and vectors changes. The mass and stiffness matrices are square symmetric matrices of the order five times the number of nodes per element. The acceleration, displacement and force vectors are of the same order.

$$\left[M^e\right]\left\{\ddot{\Delta}^e\right\}+\left[K^e\right]\left\{\Delta^e\right\}=\left\{F^e\right\} \tag{4-2}$$

where the matrices and vectors are defined as:

$$\left[M^e\right]=\begin{bmatrix} \rho_1[S] & 0 & 0 & \rho_2[S] & 0 \\ 0 & \rho_1[S] & 0 & 0 & \rho_2[S] \\ 0 & 0 & \rho_1[S] & 0 & 0 \\ \rho_2[S]^T & 0 & 0 & \rho_3[S] & 0 \\ 0 & \rho_2[S]^T & 0 & 0 & \rho_3[S] \end{bmatrix}^e \qquad \left\{\Delta^e\right\}=\begin{Bmatrix} \{u^e\} \\ \{v^e\} \\ \{w^e\} \\ \{\psi_x^e\} \\ \{\psi_y^e\} \end{Bmatrix} \tag{4-3}$$

$$\left[K^{e}\right]=\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}] \\ [K^{21}] & [K^{22}] & [K^{23}] & [K^{24}] & [K^{25}] \\ [K^{31}] & [K^{32}] & [K^{33}] & [K^{34}] & [K^{35}] \\ [K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] & [K^{45}] \\ [K^{51}] & [K^{52}] & [K^{53}] & [K^{54}] & [K^{55}] \end{bmatrix}^{e} \qquad \left\{F^{e}\right\}=\begin{Bmatrix} \{F^{1}\} \\ \{F^{2}\} \\ \{F^{3}\} \\ \{F^{4}\} \\ \{F^{5}\} \end{Bmatrix}^{e} \tag{4-4}$$

The values for the elements of the stiffness matrix are given below:

$$\begin{aligned}
\left[K^{11}\right] &= A_{11}\left[S^{xx}\right]+A_{13}\left(\left[S^{xy}\right]+\left[S^{yx}\right]\right)+A_{33}\left[S^{yy}\right] \\
\left[K^{12}\right] &= A_{12}\left[S^{xy}\right]+A_{13}\left[S^{xx}\right]+A_{23}\left[S^{yy}\right]+A_{33}\left[S^{yx}\right] \\
\left[K^{13}\right] &= [0] \\
\left[K^{14}\right] &= B_{11}\left[S^{xx}\right]+B_{13}\left(\left[S^{xy}\right]+\left[S^{yx}\right]\right)+B_{33}\left[S^{yy}\right] \\
\left[K^{15}\right] &= B_{12}\left[S^{xy}\right]+B_{13}\left[S^{xx}\right]+B_{23}\left[S^{yy}\right]+B_{33}\left[S^{yx}\right] \\
\left[K^{21}\right] &= \left[K^{12}\right]^{T} = A_{12}\left[S^{yx}\right]+A_{13}\left[S^{xx}\right]+A_{23}\left[S^{yy}\right]+A_{33}\left[S^{xy}\right] \\
\left[K^{22}\right] &= A_{22}\left[S^{yy}\right]+A_{23}\left(\left[S^{xy}\right]+\left[S^{yx}\right]\right)+A_{33}\left[S^{xx}\right] \\
\left[K^{23}\right] &= [0] \\
\left[K^{24}\right] &= B_{12}\left[S^{yx}\right]+B_{13}\left[S^{xx}\right]+B_{23}\left[S^{yy}\right]+B_{33}\left[S^{xy}\right] \\
\left[K^{25}\right] &= B_{22}\left[S^{yy}\right]+B_{23}\left(\left[S^{xy}\right]+\left[S^{yx}\right]\right)+B_{33}\left[S^{xx}\right] \\
\left[K^{31}\right] &= [0] \\
\left[K^{32}\right] &= [0] \\
\left[K^{33}\right] &= A_{44}\left[S^{yy}\right]+A_{45}\left(\left[S^{xy}\right]+\left[S^{yx}\right]\right)+A_{55}\left[S^{xx}\right] \\
\left[K^{34}\right] &= A_{45}\left[S^{y0}\right]+A_{55}\left[S^{x0}\right]
\end{aligned} \tag{4-5}$$

$$\left[K^{35}\right] = A_{44}\left[S^{y0}\right] + A_{45}\left[S^{x0}\right]$$

$$\left[K^{41}\right] = \left[K^{14}\right]^{T} = B_{11}\left[S^{xx}\right] + B_{13}\left(\left[S^{yx}\right] + \left[S^{xy}\right]\right) + B_{33}\left[S^{yy}\right]$$

$$\left[K^{42}\right] = \left[K^{24}\right]^{T} = B_{12}\left[S^{xy}\right] + B_{13}\left[S^{xx}\right] + B_{23}\left[S^{yy}\right] + B_{33}\left[S^{yx}\right]$$

$$\left[K^{43}\right] = \left[K^{34}\right]^{T} = A_{45}\left[S^{0y}\right] + A_{55}\left[S^{0x}\right]$$

$$\left[K^{44}\right] = D_{11}\left[S^{xx}\right] + D_{13}\left(\left[S^{xy}\right] + \left[S^{yx}\right]\right) + D_{33}\left[S^{yy}\right] + A_{55}[S]$$

$$\left[K^{45}\right] = D_{12}\left[S^{xy}\right] + D_{13}\left[S^{xx}\right] + D_{23}\left[S^{yy}\right] + D_{33}\left[S^{yx}\right] + A_{45}[S]$$

$$\left[K^{51}\right] = \left[K^{15}\right]^{T} = B_{12}\left[S^{yx}\right] + B_{13}\left[S^{xx}\right] + B_{23}\left[S^{yy}\right] + B_{33}\left[S^{xy}\right]$$

$$\left[K^{52}\right] = \left[K^{25}\right]^{T} = B_{22}\left[S^{yy}\right] + B_{23}\left(\left[S^{yx}\right] + \left[S^{xy}\right]\right) + B_{33}\left[S^{xx}\right]$$

$$\left[K^{53}\right] = \left[K^{35}\right]^{T} = A_{44}\left[S^{0y}\right] + A_{45}\left[S^{0x}\right]$$

$$\left[K^{54}\right] = \left[K^{45}\right]^{T} = D_{12}\left[S^{yx}\right] + D_{13}\left[S^{xx}\right] + D_{23}\left[S^{yy}\right] + D_{33}\left[S^{xy}\right] + A_{45}[S]$$

$$\left[K^{55}\right] = D_{22}\left[S^{yy}\right] + D_{23}\left(\left[S^{xy}\right] + \left[S^{yx}\right]\right) + D_{33}\left[S^{xx}\right] + A_{44}[S]$$

The values of the $[S^{\xi\eta}]$ sub-matrices are area integrals of interpolation function values and derivatives. These sub-matrices are of order n x n, and are evaluated by the following definitions (i, j = 1,2,3,...n):

$$S_{ij}^{\xi\eta} = \int_{R_e} \frac{\partial \phi_i}{\partial \xi} \frac{\partial \phi_j}{\partial \eta} dxdy, \qquad \xi, \eta = x, y$$

$$S_{ij}^{\xi 0} = \int_{R_e} \frac{\partial \phi_i}{\partial \xi} \phi_j dxdy, \quad S_{ij}^{0\xi} = \int_{R_e} \phi_i \frac{\partial \phi_j}{\partial \xi} dxdy, \qquad \xi = x, y \tag{4-6}$$

$$S_{ij} = \int_{R_e} \phi_i \phi_j dxdy$$

Note that the $\left[S^{\xi\eta}\right]$ and $\left[K^{U}\right]$ sub-matrices are not symmetric. However, since these matrices have the properties $\left[S^{\xi\eta}\right]=\left[S^{\eta\xi}\right]^T$ and $\left[K^{U}\right]=\left[K^{\Pi}\right]^T$, the total element matrices $\left[M^e\right]$ and $\left[K^e\right]$ are symmetric.

The sub-vectors of the force vector are of order n and are given by their components (i = 1,2,3,...,n):

$$F_i^{\alpha} = \int_{R_e} f_{\alpha}\phi_i dxdy + P_i, \qquad \alpha = 1,2,3,4,5 \tag{4-7}$$

For uniform transverse pressure as used in this program, $f_3 = q$ (the transverse pressure value), all other $f_{\alpha} = 0$, and P_i are the nodal contributions of the boundary force conditions along the plate boundaries.

Element Types

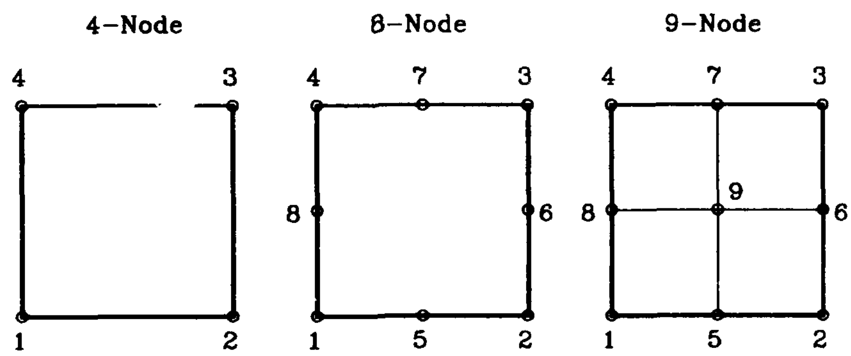

Figure 10: Element types and nodal point numbers.

The program allows the user to choose from the three element types shown in Figure 10: four-noded linear quadrilateral, eight-noded quadratic quadrilateral, and

nine-noded quadratic quadrilateral elements. The four, eight, and nine-noded elements produce element matrices of 20x20, 40x40, and 45x45, respectively.

The interpolation functions used for these elements are isoparametric and belong to the Lagrange family. The terms of the $[S^{\xi\eta}]$ matrices are found using Gauss-Legendre quadrature numerical integration. Full-integration is used for all stiffness terms except for those terms involving traverse shear coefficients (A_{44}, A_{45}, A_{55}) in which a reduced-integration scheme is used. The reduced-integration is performed to prevent shear-locking effects. J. N. Reddy presents a more complete treatment of these subjects [25].

Finite Element Procedure

For static problems, the differential equation becomes a much more simple series of linear equations because the acceleration vector is zero. In this case, the following equation applies:

$$\left[K^e\right]\left\{\Delta^e\right\} = \left\{F^e\right\} \tag{4-8}$$

In this instance, the element matrices are assembled globally into a banded matrix. Boundary conditions are then applied to the global equation and the equation is solved. The resulting vector gives the generalized displacement values in the following order:

$$\left\{\left\{u, v, w, \psi_x, \psi_y\right\}^{node1}, \left\{u, v, w, \psi_x, \psi_y\right\}^{node2}, \ldots\right\} \tag{4-9}$$

These displacement values are used to calculate strains and stresses at the Gaussian points of each element by applying the definitions of strain and constitutive relationships equation 2-3 from chapter II. The following stresses are calculated at each lamina interface through the plate thickness at the Gaussian coordinates: $\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}$.

Time-Dependent Formulation

Time-dependent problems require some extra steps in the solution because they require the solution of a second-order differential equation. This method follows that presented by Reddy [24,25]. The process requires transforming the equation shown below into a series of linear equations.

$$[M]\{\ddot{\Delta}\} + [K]\{\Delta\} = \{F\} \tag{4-10}$$

One way of transforming this equation into a solvable form is by using discrete time steps in the analysis. The method used in the program is known as the Newmark integration scheme. A time step (Δt) is chosen which yields a stable and accurate solution as described later. For each time step equation 4-9 can be expressed in the following general discretized form:

$$[\hat{K}]\{\Delta\}_{n+1} = \{\hat{F}\} \tag{4-11}$$

where the components are defined as:

$$[\hat{K}] = [K] + a_0[M]$$

$$\{\hat{F}\} = \{F\}_{n+1} + [M]\left(a_0\{\Delta\}_n + a_1\{\dot{\Delta}\}_n + a_2\{\ddot{\Delta}\}_n\right) \tag{4-12}$$

$$a_0 = \frac{1}{\beta \Delta t^2} \qquad a_1 = a_0 \Delta t \qquad a_2 = \frac{1}{2\beta} - 1$$

The values for $\{\Delta\}_n, \{\dot{\Delta}\}_n$, and $\{\ddot{\Delta}\}_n$ are the initial velocities and accelerations supplied by the program user. Once the displacement vector at the new time step $\{\Delta\}_{n+1}$ is found by solving equation 4-11, the new accelerations and velocities may calculated using the following equations:

$$\begin{aligned} \{\ddot{\Delta}\}_{n+1} &= a_0\left(\{\Delta\}_{n+1} - \{\Delta\}_n\right) - a_1\{\dot{\Delta}\}_n - a_2\{\ddot{\Delta}\}_n \\ \{\dot{\Delta}\}_{n+1} &= \{\dot{\Delta}\}_n + \left[(1-\alpha)\{\ddot{\Delta}\}_n + \alpha\{\ddot{\Delta}\}_{n+1}\right]\Delta t \end{aligned} \tag{4-13}$$

The α and β variables are parameters used in the Newmark scheme to create a stable and accurate integration solution. There are two widely used pairs for these parameters which guarantee stability in the analysis.

$$\begin{aligned} &\text{Linear acceleration method: } \alpha = 1/2,\ \beta = 1/6 \\ &\text{Constant acceleration method: } \alpha = 1/2,\ \beta = 1/4 \end{aligned} \tag{4-14}$$

These two pairs guarantee stability in the time integration scheme, but they do not necessarily provide accuracy. In order to obtain accurate results, the time step must be chosen appropriately. In most cases, shorter time steps yield more accurate approximations than longer ones. The following formula provides one way to choose an adequate time step.

$$\Delta t \leq \tfrac{1}{4} d^2 \sqrt{\rho_1 D} \qquad (4\text{-}15)$$

In this formula, d is the minimum distance between any two nodal points, ρ_1 is the transverse inertial term defined in equation 2-12, and D is the lesser of the two bending stiffness terms D_{11} and D_{22} defined in equation 2-10. Care should be given in choosing too small a time step because of the computational cost of that choice.

With these additional steps, the procedure presented for the static case is followed. This procedure requires solving a set of linear equations for the displacement vector and computing then new velocity and acceleration vectors for each time step. For this reason, the computing expense can become quite high.

Boundary Conditions

Equation 3-37 dictates the boundary condition pairs which must be specified along the plate edges. When choosing boundary conditions for a plate problem, one of each pair must be specified at each nodal point.

$$N_n \text{ or } u_n, \quad N_{ns} \text{ or } u_s, \quad Q_n \text{ or } w, \quad M_n \text{ or } \psi_n, \quad M_{ns} \text{ or } \psi_s \qquad (4\text{-}16)$$

The following list describes the applicable force and displacement values for some commonly used boundary conditions.

1. Simply-Supported Edge

$$N_n = 0, \; N_{ns} = 0, \; w = 0, \; M_n = 0, \; \psi_s = 0 \qquad (4\text{-}17)$$

2. Hinged Edge - Free in the normal direction

$$N_n = 0,\ u_s = 0,\ w = 0,\ M_n = 0,\ \psi_s = 0 \tag{4-18}$$

3. Hinged Edge - Free in the tangential direction

$$u_n = 0,\ N_{ns} = 0,\ w = 0,\ M_n = 0,\ \psi_s = 0 \tag{4-19}$$

4. Clamped Edge

$$u_n = 0,\ u_s = 0,\ w = 0,\ \psi_n = 0,\ \psi_s = 0 \tag{4-20}$$

5. Free Edge

$$N_n = 0,\ N_{ns} = 0,\ Q_n = 0,\ M_n = 0,\ M_{ns} = 0 \tag{4-21}$$

6. Line of Symmetry (for symmetrical finite element problems)

$$u_n = 0,\ N_{ns} = 0,\ Q_n = 0,\ \psi_n = 0,\ M_{ns} = 0 \tag{4-22}$$

Program Information

For implementation of the displacement and force boundary conditions in COMPLATE, the following information is necessary. By default, all generalized displacements are assumed to be free to move and all forces are assumed to be zero. The displacement and force vectors each have five times the number of nodes entries. The ordering of the displacement and force vectors are shown below:

$$\begin{aligned}
&\text{Displacement Vector} \rightarrow \left\{\left\{u, v, w, \psi_x, \psi_y\right\}^{\text{node1}}, \left\{u, v, w, \psi_x, \psi_y\right\}^{\text{node2}}, \ldots\right\} \\
&\text{Force Vector} \rightarrow \left\{\left\{N_x, N_y, Q, M_x, M_y\right\}^{\text{node1}}, \left\{N_x, N_y, Q, M_x, M_y\right\}^{\text{node2}}, \ldots\right\}
\end{aligned} \tag{4-23}$$

To change a boundary condition from the default conditions, the user must supply the position of the displacement or force in its respective vector and the value

of that condition. For example, to specify a displacement of 10 in the y-direction on node 3, the user would give 12 for its position in the vector and 10 for its value.

Summary

The basic theory behind the development of COMPLATE has been presented in the last two chapters. This program is written in the FORTRAN-77 standard for use on microcomputers although the code is generic and may be used on larger systems. Appendices B and C provide more information on the program including user information, sample program input and output data files, and documented source code.

CHAPTER V

NUMERICAL EXAMPLES

Overview

In this chapter several numerical examples are presented to validate the computer code. Although the computer program can analyze more complicated composite plate shapes and laminate lay-ups, the following cases are chosen because other solution methods are available for these types of composite plates and loadings. Two static and one dynamic plate problems are considered. The static problems are compared against analytical solutions using the Navier series. The dynamic problem is compared against a solution given in the literature.

The Navier series solution method is presented in Appendix A for the two plate problems in this chapter. The Navier series is used to find an exact solution to both the classical plate theory (CPT) and first-order shear deformation theory (FSDT) plate equations. The CPT solution demonstrates the importance of shear deformation in the analysis.

The plate used in all three cases is square with sides of length, a, as shown in figure 11. The loading condition for all three cases is a uniform transverse pressure applied over the area of the plate surface. The laminate stacking sequence and the boundary conditions change for the three cases in order to utilize more of the computer code.

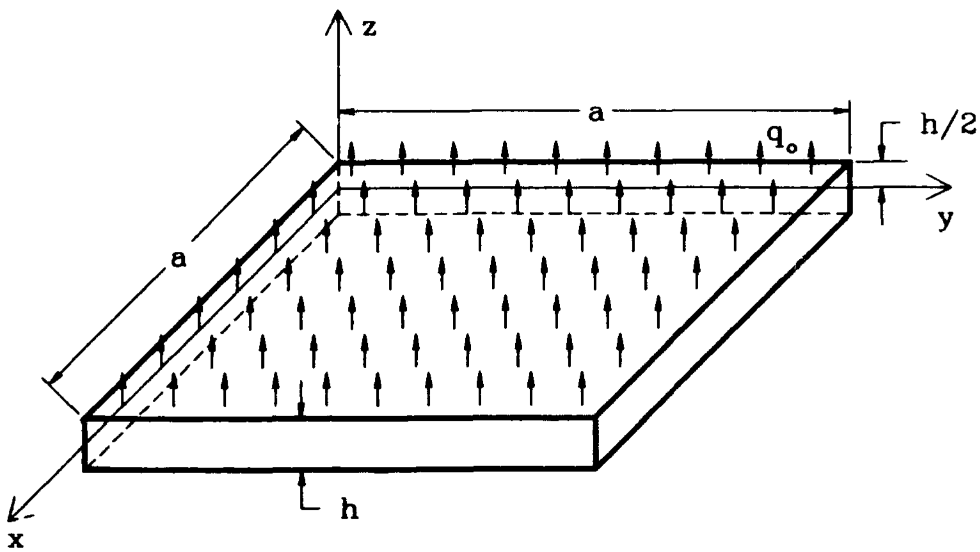

Figure 11: Plate dimensions and loading for Cases 1, 2, and 3.

For the two static cases, Cases 1 and 2, the following material properties are used which are representative of composite materials. These particular properties were introduced in 1969 [15] and have been used by many other author's since.

$$\begin{array}{ll} E_1 = 25\times10^6 \text{ psi} & E_2 = 1\times10^6 \text{ psi} \\ G_{12} = G_{13} = 0.5\times10^6 \text{ psi} & G_{23} = 0.2\times10^6 \text{ psi} \\ \nu_{12} = 0.25 & k_{sc} = 5/6 \end{array} \qquad (5\text{-}1)$$

Case 1: Symmetric Specially-Orthotropic Square Plate under Uniform Transverse Pressure (Simply-Supported)

For this case, a symmetric specially-orthotropic laminate is chosen for its laminate property features. The definition of such a laminate is one which exhibits the following laminate properties.

$$A_{13} = A_{23} = A_{45} = B_{ij} = D_{13} = D_{23} = 0 \tag{5-2}$$

This type of laminate behaves like an orthotropic material for general response purposes because the laminate stiffness matrix contains the same zero-valued terms as an orthotropic material. All the bending-twisting coupling terms vanish. For the calculation of interlaminar stresses, the lamina constitutive relationships are required. Therefore, a program which is able to analyze orthotropic plates is insufficient. Because the B matrix is zero and there is no applied in-plane loading, in-plane displacements decouple from the transverse deflection, w, and the rotations, ψ_x and ψ_y. This results in no in-plane displacements of the laminate mid-plane (u = v = 0).

For this case, the laminate stacking sequence is [0/90/0] where all three layers have equal thicknesses, namely h/3. This type of lay-up is called a cross-ply because the laminate is constructed of only 0 and 90 layers.

This plate is assumed to have simply-supported boundary conditions along all the edges. This results in the following boundary conditions (ignoring in-plane boundary conditions).

$$\begin{aligned} &w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0 \\ &\psi_x(x,0) = \psi_x(x,b) = \psi_y(0,y) = \psi_y(a,y) = 0 \\ &M_x(0,y) = M_x(a,y) = M_y(x,0) = M_y(x,b) = 0 \end{aligned} \tag{5-3}$$

This case is used to evaluate the effectiveness of each element type compared to the exact analytical solution so all three element types are used in the analysis. Also the effect of shear deformation is observed by using three length to thickness ratios (a/h = 100,10,4).

Because of this problem's symmetry, only one quarter of the plate is needed for the analysis. This results in the meshes shown in figure 12.

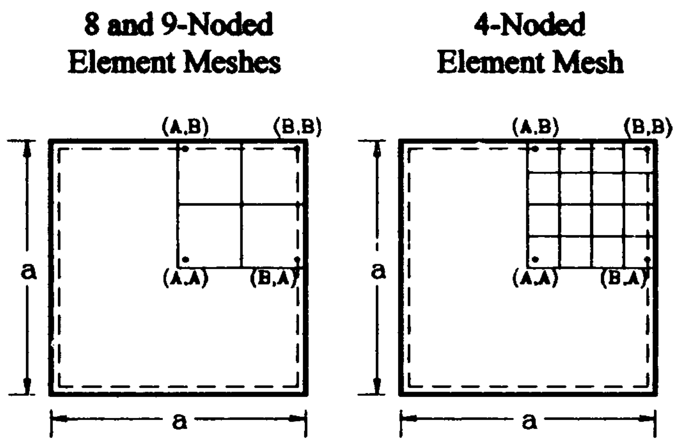

Figure 12: Geometry of FEM mesh for Cases 1 and 3.

The results of this analysis are shown in Table 1. The deflection and stresses are non-dimensionalized by the definitions given in equation 5-4. The stresses shown in the table are calculated at Gaussian points with coordinates defined by A and B in equation 5-5.

$$\overline{w} = w\left(\frac{a}{2},\frac{a}{2},0\right)\cdot\frac{h^3E_2}{q_0a^4}\cdot 100$$

$$\overline{\sigma}_x = \sigma_x\left(A,A,\frac{h}{2}\right)\cdot\frac{h^2}{q_0a^2}$$

$$\overline{\sigma}_y = \sigma_y\left(A,A,\frac{h}{6}\right)\cdot\frac{h^2}{q_0a^2}\ \text{in layer 2 } (90^\circ) \tag{5-4}$$

$$\overline{\tau}_{xy} = \tau_{xy}\left(B,B,\frac{h}{2}\right)\cdot\frac{h^2}{q_0a^2}$$

$$\bar{\tau}_{xz} = \tau_{xz}(B,A)\cdot\frac{h}{q_0 a} \text{ in layer 3 } (0^\circ)$$

$$\bar{\tau}_{yz} = \tau_{yz}(A,B)\cdot\frac{h}{q_0 a} \text{ in layer 2 } (90^\circ)$$

A = 0.55283 x a (for 2x2Q), 0.56250 x a (for 4x4L)

B = 0.94717 x a (for 2x2Q), 0.93750 x a (for 4x4L) (5-5)

The terms in the "Method" column of Table 1 refer to following methods of analysis:

COMPLATE Results:
- 2x2Q9- 4 element mesh using 9-noded quadratic elements.
- 2x2Q8- 4 element mesh using 8-noded quadratic elements.
- 4x4L4- 16 element mesh using 4-noded linear elements.

Other method results:
- CPT- Navier series solution using classical plate theory, Appendix A.
- FSDT- Navier series solution using first-order shear deformation theory, Appendix A
- HSDT- Center deflection solution by Reddy using higher-order shear deformation theory[22].

Navier series solutions utilized 49 terms (m,n = 1...49) - see Appendix A.

The results in Table 1 show that COMPLATE produces results close to those of the Navier solution using FSDT for coarse meshes using both 8 and 9-noded quadratic elements. The 4-noded linear element appears to be less accurate. Also note that the non-dimensionalized deflection, w, is much higher for the cases a/h=10 and 4 than predicted by CPT.

Table 1: Comparison of maximum deflection and stresses for Case 1 [0/90/0].

	Method	$\overline{w}$	$\overline{\sigma}_x$	$\overline{\sigma}_y$	$\overline{\tau}_{xy}$	$\overline{\tau}_{xz}$	$\overline{\tau}_{yz}$
	CPT	0.6660	0.7908	0.1935	-0.03944	0	0
a/h=100	2x2Q9	0.6704	0.7931	0.1949	-0.03861	-0.7036	-0.2049
	2x2Q8	0.6707	0.7933	0.2015	-0.03757	-0.7118	-0.1999
	4x4L4	0.6660	0.7752	0.1960	-0.03631	-0.6968	-0.1967
	HSDT[22]	0.6705					
	FSDT	0.6697	0.7905	0.1948	-0.03929	-0.7020	-0.2020
a/h=10	2x2Q9	1.0234	0.7577	0.3076	-0.04555	-0.6865	-0.2336
	2x2Q8	1.0211	0.7575	0.3078	-0.04516	-0.6872	-0.2340
	4x4L4	1.0276	0.7386	0.3096	-0.04332	-0.6758	-0.2247
	HSDT[22]	1.0900					
	FSDT	1.0219	0.7556	0.3066	-0.04657	-0.6823	-0.2294
a/h=4	2x2Q9	2.6630	0.6419	0.6513	-0.06588	-0.6143	-0.3275
	2x2Q8	2.6559	0.6419	0.6513	-0.06544	-0.6143	-0.3278
	4x4L4	2.7025	0.6238	0.6494	-0.06300	-0.6020	-0.3198
	HSDT[22]	2.9091					
	FSDT	2.6595	0.6408	0.6494	-0.06725	-0.6104	-0.3230

Case 2: Angle-Ply Square Plate under Uniform Transverse Pressure (Hinged)

For this case, an angle-ply laminate is chosen for its laminate property features. The definition of such a laminate is one with the stacking sequence $[+\theta/-\theta]_n$ where n is some integral multiple. This type of laminate has the following laminate property simplifications:

$$A_{13} = A_{23} = A_{45} = B_{11} = B_{12} = B_{22} = B_{33} = D_{13} = D_{23} = 0 \tag{5-6}$$

The only bending-twisting coupling terms appear in the B matrix. This case is able to test the program's coupling effect calculations.

For this case, the laminate stacking sequence is $[45/-45]_2$ where all four layers have equal thicknesses, namely h/4. This plate is assumed to have hinged edges with freedom in the tangential direction along the plate boundaries. This results in the following boundary conditions:

$$\begin{aligned}
&w(0,y) = w(a,y) = w(x,0) = w(x,a) = 0\\
&u(0,y) = u(a,y) = v(x,0) = v(x,a) = 0\\
&\psi_x(x,0) = \psi_x(x,a) = \psi_y(0,y) = \psi_y(a,y) = 0\\
&M_y(x,0) = M_y(x,a) = M_x(0,y) = M_x(a,y) = 0\\
&N_x(x,0) = N_x(x,a) = N_y(0,y) = N_y(a,y) = 0
\end{aligned} \tag{5-7}$$

Since all three element types were evaluated in the preceding case, only the 9-noded quadratic element is used for this case. B cause the laminate stacking sequence is not symmetric about its mid-plane, the same plate symmetry as in case 1

does not exist for this case. Therefore a full 4x4 mesh of the entire plate is used as shown in Figure 13.

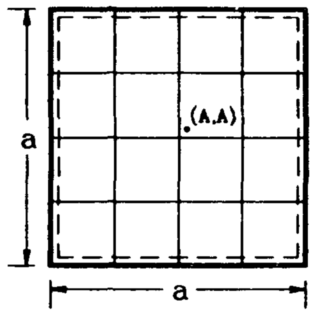

Figure 13: Geometry of FEM mesh for Case 2.

The results of this analysis are shown in Table 2. The deflection and stresses are non-dimensionalized by the definitions given in equation 5-8. The stresses shown in the table are calculated at the Gaussian points coordinates defined by A.

$$
\begin{aligned}
\overline{w} &= w\left(\frac{a}{2},\frac{a}{2},0\right)\cdot\frac{h^3E_2}{q_0a^4}\cdot 100 \\
\overline{\sigma}_x &= \sigma_x\left(A,A,\frac{h}{2}\right)\cdot\frac{h^2}{q_0a^2} \\
\overline{\sigma}_y &= \sigma_y\left(A,A,\frac{h}{2}\right)\cdot\frac{h^2}{q_0a^2} \\
\overline{\tau}_{xy} &= \tau_{xy}\left(A,A,\frac{h}{2}\right)\cdot\frac{h^2}{q_0a^2} \\
\overline{\tau}_{yz} &= \tau_{yz}\left(A,A,\frac{h}{2}\right)\cdot\frac{h}{q_0a}
\end{aligned}
\qquad (5\text{-}8)
$$

$$\bar{\tau}_{xz} = \tau_{xz}(A,A) \cdot \frac{h}{q_0 a}$$

$$A = \quad 0.55283 \text{ x a} \tag{5-9}$$

4x4Q9 refers to results from COMPLATE. The other methods are defined in the same way as in case 1. In addition to tabular results, the stresses are plotted through the thickness to show a representation of interlaminar stresses in Figures 14 to 16.

The results in Table 2 show that the computer program provides adequate accuracy compared to the Navier solution of FSDT. Also note that the non-dimensionalized deflection, w, is much higher for the cases a/h=10 and 4 than predicted by CPT.

Table 2: Comparison of maximum deflection and stresses for Case 2 $[\pm 45]_2$.

	Method	$\overline{W}$	$\overline{\sigma}_x$	$\overline{\sigma}_y$	$\overline{\tau}_{xy}$	$\overline{\tau}_{xz}$	$\overline{\tau}_{yz}$
	CPT	0.4408	0.2040	0.2040	-0.1774	0	0
a/h=100	4x4Q9	0.4439	0.2041	0.2041	-0.1776	-0.01783	-0.01783
	FSDT	0.4433	0.2039	0.2039	-0.1773	-0.01789	-0.01789
a/h=10	4x4Q9	0.6925	0.2011	0.2011	-0.1751	-0.01773	-0.01773
	FSDT	0.6917	0.2007	0.2007	-0.1746	-0.01788	-0.01788
a/h=4	4x4Q9	2.0186	0.1986	0.1986	-0.1731	-0.01763	-0.01763
	FSDT	2.0164	0.1977	0.1977	-0.1722	-0.01784	-0.01784

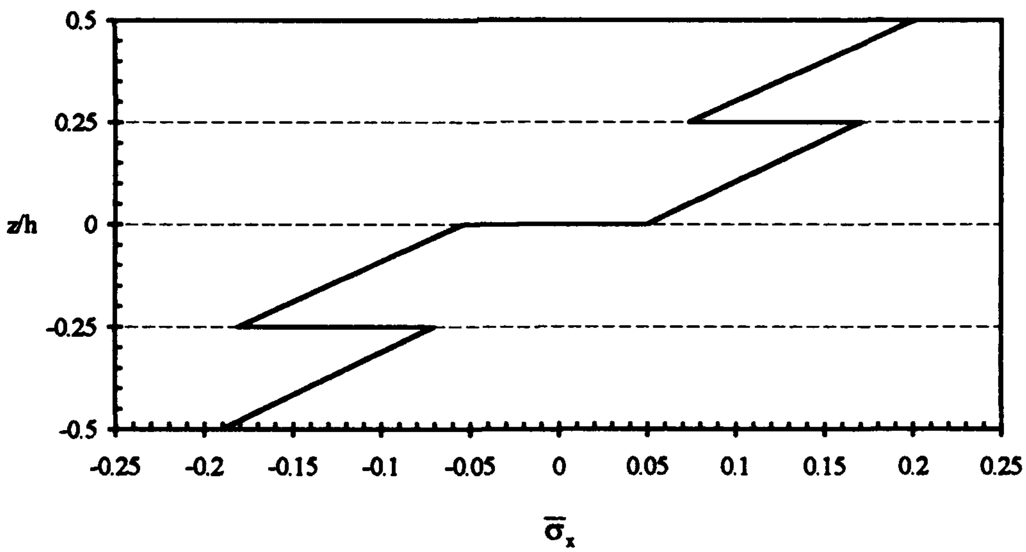

Figure 14: Normal stress, $\overline{\sigma}_x$, through the plate thickness near the plate center for Case 2 $[\pm 45]_2$.

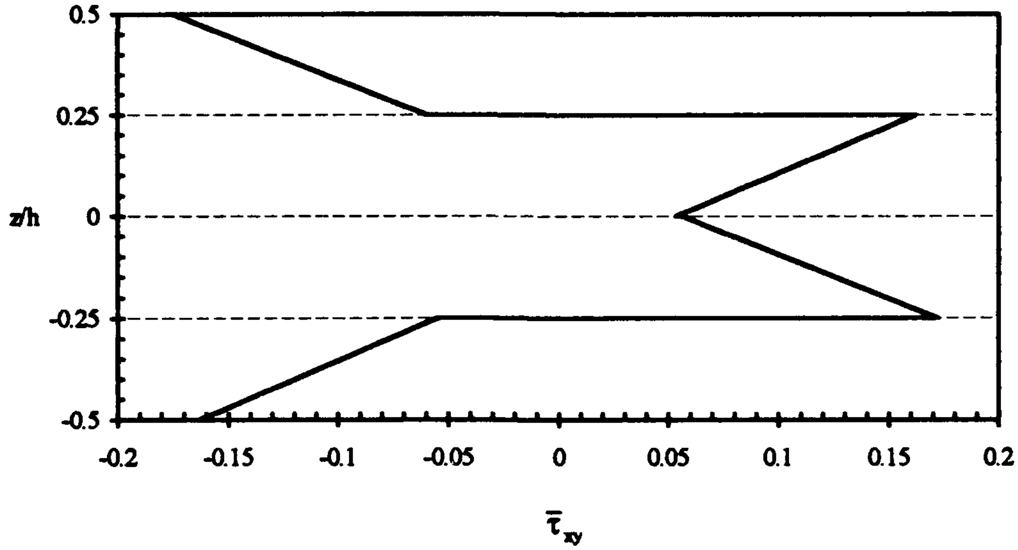

Figure 15: In-plane shear stress, $\bar{\tau}_{xy}$, through the plate thickness near the plate center for Case 2 $[\pm 45]_2$.

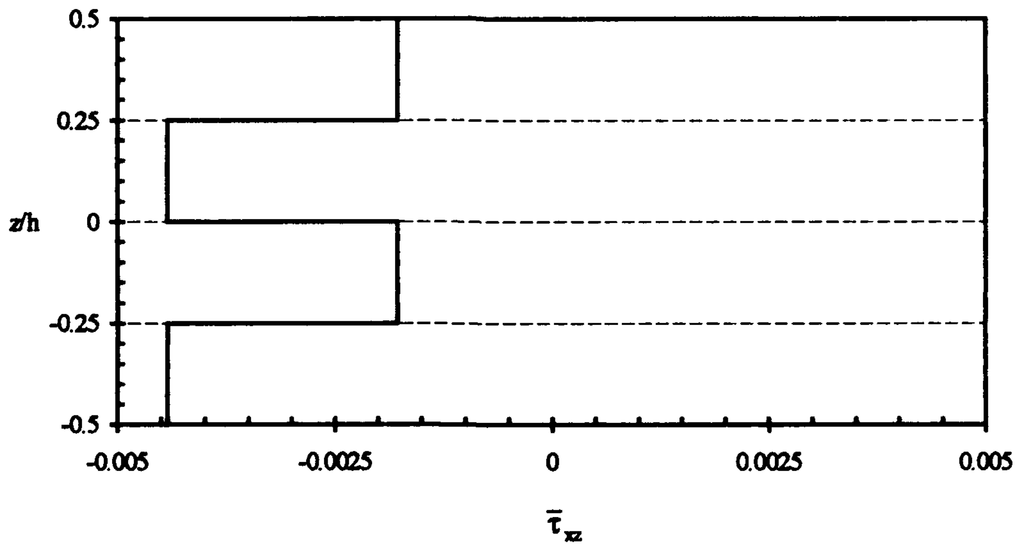

Figure 16: Transverse shear stress, $\bar{\tau}_{xz}$, through the plate thickness near the plate center for Case 2 $[\pm 45]_2$.

Case 3: Orthotropic Square Plate under Suddenly Applied Uniform Transverse Pressure (Simply-Supported)

In order to validate the dynamic analysis routines, a simple orthotropic laminate is chosen for this case. Results presented by Reddy [23] are used in comparison. The material properties and time parameters used in this analysis are also taken from [23]. The following material properties are used.

$$
\begin{aligned}
&E_1 = 52.5\times10^6 \; {}^{N}\!/_{cm^2} && E_2 = 2.1\times10^6 \; {}^{N}\!/_{cm^2} \\
&G_{12} = G_{13} = G_{23} = 1.05\times10^6 \; {}^{N}\!/_{cm^2} && \\
&\nu_{12} = 0.25 && \rho = 8\cdot10^{-6} \; {}^{N\,sec^2}\!/_{cm^4} \\
&k_{\infty} = {}^{5}\!/_{6} &&
\end{aligned}
\tag{5-10}
$$

and the time parameters for the Newmark scheme (constant-average acceleration) are:

$$\Delta t = 5\ \mu sec,\ \alpha = 0.5,\ \beta = 0.25 \tag{5-11}$$

Since only one layer exists, the stacking sequence is [0]. The plate edges are simply-supported and a uniform pressure is suddenly applied at the initial time step which remains steady throughout the analysis. The boundary conditions are the same as in Case 1. Symmetry is again utilized and the mesh is the same as the one used with the 2x2Q9 elements in Case 1. Since a direct comparison is made with the

literature, the results are not non-dimensionalized. The plate is square with the following geometric parameters:

$$a = 25 \text{ cm}, \; h = 5 \text{ cm} \tag{5-12}$$

Table 3 shows the results of the center deflection and the stress at the Gaussian point nearest to the plate center (A,A) over time compared to the results presented by Reddy. Since Reddy used a similar procedure, the results coincide exactly except for a couple of what appear to be typographical errors in his article. This correlation validates the dynamic routines of the program.

Table 3: Dynamic response, deflection (w) and normal stress (σ_x), of a square orthotropic plate under pulse uniform transverse pressure, Case 3.

time	$w \cdot 10^3$ (cm)		σ_x (N/cm^2)	
(μsec)	Reddy[23]	2x2Q9	Reddy[23]	2x2Q9
10	0.0079	0.007963	2.986	2.986
20	0.0398	0.03985	24.64	24.64
40	0.1939	0.1939	132.2	132.2
60	0.4303	0.4303	282.1	282.1
80	0.5531	0.5531	359.3	359.3
100	0.5264	0.5264	349.7	349.7
120	0.3705	0.3705	345.4	245.4
140	0.1779	0.1779	115.1	115.1
160	0.0353	0.03533	22.0	22.0
180	-0.0395	-0.03946	-20.97	-20.97
200	0.1105	0.1105	73.61	73.61
220	0.3296	0.3296	214.1	214.1
240	0.4781	0.4781	316.8	316.8
260	0.5548	0.5548	368.9	368.9
280	0.4797	0.4797	314.5	314.5
300	0.2006	0.3006	194.9	194.9
320	0.0840	0.08402	59.38	59.38
340	-0.0302	-0.03020	-18.53	-18.53
360	0.0459	0.04587	28.57	28.57

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The primary objective of this work was to develop a program to analyze general laminated composite plates under static and dynamic conditions using first-order shear deformation theory. Several composite plate cases were used to validate the computer code by comparing the results against other solution methods and results found in literature. The following conclusions may be drawn from the results:

1. Based on the three cases, the program appears to correctly apply the first-order shear deformation theory of composite plates to the finite element method.

2. Of the three element types used in the program, the 8 and 9-noded quadratic elements give the best results with neither one showing clear superiority. The 4-noded linear element appears to give less than adequate results compared to the two previous element types.

3. The deflections predicted by the computer program show that shear deformation effects can be significant beyond the usual range for isotropic materials ($a/h > 10$). This is a clear demonstration for the need of a shear deformation theory for the analysis of all but very thin composite plates.

4. Several authors have presented a complete treatment of classical plate theory for composite plates. Because of length restrictions in published articles, FSDT is often presented in abbreviated form. Current composite textbooks give only

a cursory treatment of the first-order shear deformation theory and neglect to detail the full importance of shear deformation in composite plates. This work represents a full presentation of this theory combined into a single source.

Recommendations

Based on research in composite plate theories and the development of the computer program, the following modifications and future research are anticipated:

1. This program should be used in advanced composite classes to give students insight on the importance of shear deformation in composite plates and to gain a better understanding of the finite element formulation of composite plate theories.

2. This program could be used in conjunction with composite testing if funds are available for future work. The program could be used in developing the size and stacking sequence of the composite test specimen, and also the program results could be directly compared with test results.

3. Because the environmental responses of composites are significant, thermal and hygral (moisture) effects should be incorporated in future revisions of this program.

4. The program provides easy application of a uniform transverse pressure. Other types of transverse loading require a significant amount of work on the user's part. A subroutine should be developed to apply more general types of loading functions, $q = q(x,y)$.

5. This program provides a good starting point to develop a program based on higher-order shear deformation theories of composite plates. Because of the

additional degrees of freedom required in these theories, a microcomputer-based program appears to be prohibitive at present. However, a program can be developed on a larger system or in the future on smaller computers as memory management problems are solved.

APPENDIX A

NAVIER SERIES SOLUTIONS

Navier proposed using a double Fourier series to solve certain differential equations. This type of series was applied to the solution of plate governing equations with particular shapes and boundary conditions. Because this solution satisfies the plate governing equations and meets the applicable boundary conditions, it is viewed as an exact solution of the plate theory used in the analysis. Since the series is infinite, it is impossible to utilize all the terms. However taking enough terms in the solution (m,n < 50), the solution converges to the significant figures presented as results in chapter V.

For this analytical solution, general transverse pressure, q(x,y), on a rectangular plate can be applied by use of the following Navier series:

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A-1}$$

The loading terms (Q_{mn}) can be found by integrating equation A-1, and then solving for Q_{mn}. This yields the following result:

$$Q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dydx \tag{A-2}$$

Since the cases presented in the results utilize a uniformly applied transverse pressure ($q(x,y) = q_o$), it is convenient to define the loading terms for this condition. For a uniform transverse pressure, these terms result in the following expression:

$$Q_{mn} = \frac{4q_o}{mn\pi^2}(1-\cos m\pi)(1-\cos n\pi) \qquad \text{(A-3)}$$

CASE 1: SYMMETRIC SPECIALLY-ORTHOTROPIC LAMINATES

Symmetric specially-orthotropic plates are often used to compare results because they simplify the governing plate equations. For plates with simply-supported edges, the analysis process can be accomplished by use of a Navier solution. This type of plate is defined as being symmetric with respect to the laminate mid-plane and whose bending-twisting coupling terms vanish. This translates to plate lay-ups with the following material property simplifications:

$$A_{13} = A_{23} = B_{ij} = D_{13} = D_{23} = A_{45} = 0 \qquad \text{(A-4)}$$

Classical Plate Theory Solution:

The CPT solution ignores shear deformation and is therefore independent of thickness effects. This solution is used as a baseline case for very thin plates. Whitney [33] provides a more complete treatment of this method. The following boundary conditions are used for the CPT solution of the simply-supported rectangular plate:

$$
\begin{aligned}
&w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0 \\
&\frac{\partial w}{\partial x}(x,0) = \frac{\partial w}{\partial x}(x,b) = \frac{\partial w}{\partial y}(0,y) = \frac{\partial w}{\partial y}(a,y) = 0 \\
&M_x(0,y) = M_x(a,y) = M_y(x,0) = M_y(x,b) = 0
\end{aligned}
\tag{A-5}
$$

With the given material property simplifications, the governing CPT equations reduce to the following:

$$
D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial x^4} = q \tag{A-6}
$$

The deflection function is chosen to satisfy the plate boundary conditions and the simplified CPT governing equations and is given as::

$$
w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A-7}
$$

By applying this function to the CPT equation, the following relation for the constant W_{mn} is obtained (with R = a/b):

$$
\begin{aligned}
&W_{mn} = \frac{a^4}{\pi^4}\frac{Q_{mn}}{D_{mn}} \\
&\quad \text{where } D_{mn} = D_{11}m^4 + 2(D_{12} + 2D_{66})(mnR)^2 + D_{22}(nR)^4
\end{aligned}
\tag{A-8}
$$

The in-plane stresses in the k^{th} layer are defined by the following equations:

$$
\sigma_x^{(k)}(x,y,z) = \frac{a^2}{\pi^2} z \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{Q_{mn}}{D_{mn}} \left(\overline{Q}_{11}^{(k)} m^2 + \overline{Q}_{12}^{(k)} n^2 R^2\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}
$$

$$\sigma_y^{(k)}(x,y,z) = \frac{a^2}{\pi^2} z \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{Q_{mn}}{D_{mn}} \left(\overline{Q}_{12}^{(k)} m^2 + \overline{Q}_{22}^{(k)} n^2 R^2\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A-9}$$

$$\tau_{xy}^{(k)}(x,y,z) = -2\frac{a^2 R}{\pi^2} \overline{Q}_{33}^{(k)} z \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{mnQ_{mn}}{D_{mn}} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

Note that CPT assumes that the transverse shear strains are negligible, and therefore the transverse shear stresses are zero.

First-Order Shear deformation theory Navier solution:

Assume the following Navier Series displacements functions:

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\psi_x(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Psi x_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A-10}$$

$$\psi_y(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Psi y_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

These assumed displacement functions meet the following simply-supported boundary conditions on the plate:

$$\begin{aligned} &w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0 \\ &\psi_x(x,0) = \psi_x(x,b) = \psi_y(0,y) = \psi_y(a,y) = 0 \\ &M_x(0,y) = M_x(a,y) = M_y(x,0) = M_y(x,b) = 0 \end{aligned} \tag{A-11}$$

These assumed displacement fields can be substituted into the governing FSDT equations (3-40 to 3-43). Since the B matrix terms are zero, the in-plane

displacements (u,v) become uncoupled from the other generalized displacements (w, ψ_x, ψ_y). Under static conditions with only transverse pressure applied, equations 3-40 to 3-43 reduce to the following equations:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} \begin{Bmatrix} \Psi x_{mn} \\ \Psi y_{mn} \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \end{Bmatrix} \tag{A-12}$$

$$\begin{aligned} L_{11} &= D_{11}\alpha_m^2 + D_{33}\beta_n^2 + A_{55} \\ L_{12} &= (D_{12} + D_{33})\alpha_m\beta_n \\ L_{13} &= A_{55}\alpha_m \\ L_{22} &= D_{33}\alpha_m^2 + D_{22}\beta_n^2 + A_{44} \\ L_{23} &= A_{44}\beta_n \\ L_{33} &= A_{55}\alpha_m^2 + A_{44}\beta_n^2 \end{aligned} \tag{A-13}$$

where $\alpha_m = \dfrac{m\pi}{a}$ and $\beta_n = \dfrac{n\pi}{b}$

The solution of this equation yields the constants $(W_{mn}, \Psi x_{mn}, \Psi y_{mn})$ which can be substituted into equation A-10 to determine the deflection and rotations of any point on the plate mid-plane. The plate stresses are defined as:

$$\sigma_x^{(k)}(x,y,z) = -z\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\overline{Q}_{11}^{(k)}\Psi x_{mn}\alpha_m + \overline{Q}_{12}^{(k)}\Psi y_{mn}\beta_n\right)\sin\alpha_m x \sin\beta_n y$$

$$\sigma_y^{(k)}(x,y,z) = -z\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\overline{Q}_{12}^{(k)}\Psi x_{mn}\alpha_m + \overline{Q}_{22}^{(k)}\Psi y_{mn}\beta_n\right)\sin\alpha_m x \sin\beta_n y$$

$$\tau_{xy}^{(k)}(x,y,z) = z\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{33}^{(k)}\left(\Psi x_{mn}\beta_n + \Psi y_{mn}\alpha_m\right)\cos\alpha_m x \cos\beta_n y \tag{A-14}$$

$$\tau_{yz}^{(k)}(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{44}^{(k)}(\Psi y_{mn}+W_{mn}\beta_n)\sin\alpha_m x\cos\beta_n y$$
$$+\overline{Q}_{45}^{(k)}(\Psi x_{mn}+W_{mn}\alpha_n)\cos\alpha_m x\sin\beta_n y$$

$$\tau_{yz}^{(k)}(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{45}^{(k)}(\Psi y_{mn}+W_{mn}\beta_n)\sin\alpha_m x\cos\beta_n y$$
$$+\overline{Q}_{55}^{(k)}(\Psi x_{mn}+W_{mn}\alpha_n)\cos\alpha_m x\sin\beta_n y$$

CASE 2: ANGLE-PLY LAMINATES $[\pm\theta]_n$

Angle-ply laminated plates are also used to compare results because they allow the governing plate equations to be simplified, but still include bending-stretching coupling. With certain boundary conditions, the analysis process does not require a numerical solution. Angle-ply laminates are defined by a laminate stacking sequence with fiber orientations alternating between $+\theta$ to $-\theta$ among consecutive layers. The code for such laminates is $[\pm\theta]_n$ or $[+\theta/-\theta]_n$, where n is some integer multiple. This translates to plate lay-ups with the following material property simplifications:

$$A_{13}=A_{23}=B_{11}=B_{12}=B_{22}=B_{33}=D_{13}=D_{23}=A_{45}=0 \qquad \text{(A-15)}$$

For this analytical solution, general transverse pressure on a square plate is applied in the same manner as given in equations A-1 to A-3.

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \text{(A-16)}$$

Classical Plate Theory Solution:

The CPT solution ignores shear deformation and is independent of the plate thickness effects. This solution is used as a baseline case for very thin plates. Whitney [33] provides a more complete treatment of this solution. The plate considered here is rectangular with hinged-edges (free in the tangential direction). In this case, the following boundary conditions apply along the edges:

$$
\begin{aligned}
&w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0 \\
&u(0,y) = u(a,y) = v(x,0) = v(x,b) = 0 \\
&\frac{\partial w}{\partial y}(0,y) = \frac{\partial w}{\partial y}(a,y) = \frac{\partial w}{\partial x}(x,0) = \frac{\partial w}{\partial x}(x,b) = 0 \\
&N_{xy}(0,y) = N_{xy}(a,y) = N_{xy}(x,0) = N_{xy}(x,b) = 0 \\
&M_x(0,y) = M_x(a,y) = M_y(x,0) = M_y(x,b) = 0
\end{aligned}
\qquad \text{(A-17)}
$$

With the material simplifications of angle-ply laminates, the governing CPT differential equations become:

$$
\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix}
\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q \end{Bmatrix}
\qquad \text{(A-18)}
$$

where:

$$
\begin{aligned}
K_{11} &= A_{11}\frac{\partial^2}{\partial x^2} + A_{33}\frac{\partial^2}{\partial y^2} \\
K_{12} &= (A_{12} + A_{33})\frac{\partial^2}{\partial x \partial y} \\
K_{13} &= 3B_{13}\frac{\partial^3}{\partial x^2 \partial y} + B_{23}\frac{\partial^3}{\partial y^3} \\
K_{22} &= A_{33}\frac{\partial^2}{\partial x^2} + A_{22}\frac{\partial^2}{\partial y^2}
\end{aligned}
\qquad \text{(A-19)}
$$

$$K_{23} = B_{13}\frac{\partial^3}{\partial x^3} + 3B_{23}\frac{\partial^3}{\partial x \partial y^2}$$

$$K_{33} = D_{11}\frac{\partial^4}{\partial x^4} + 2(D_{12} + 2D_{33})\frac{\partial^4}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4}{\partial y^4}$$

The deflection functions, equation A-20, are chosen to satisfy the plate boundary conditions and the above simplified CPT governing equation:

$$u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$v(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \text{(A-20)}$$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

By substituting these displacement functions into the CPT equations, the following matrix equation results:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \end{Bmatrix} \qquad \text{(A-21)}$$

where:

$$\begin{aligned}
L_{11} &= A_{11}\alpha_m^2 + A_{33}\beta_n^2 \\
L_{12} &= (A_{12} + A_{33})\alpha_m\beta_n \\
L_{13} &= -3B_{13}\alpha_m^2\beta_n - B_{23}\beta_n^3 \\
L_{22} &= A_{33}\alpha_m^2 + A_{22}\beta_n^2 \\
L_{23} &= -B_{13}\alpha_n^3 - 3B_{23}\alpha_m\beta_n^2 \\
L_{33} &= D_{11}\alpha_m^4 + 2(D_{12} + 2D_{33})\alpha_m^2\beta_n^2 + D_{22}\beta_n^4
\end{aligned} \qquad \text{(A-22)}$$

and $\alpha_m = \frac{m\pi}{a}$ and $\beta_n = \frac{n\pi}{b}$.

The solution of these equations yields the constants (U_{mn}, V_{mn}, W_{mn}) which can be substituted into equation A-20 to determine the deflection and rotations of any point on the plate mid-plane. These constants are also utilized to calculate the plate stresses. The in-plane stresses in the kth layer are defined by the following equations:

$$\begin{aligned}\sigma_x^{(k)}(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[&\left(\overline{Q}_{11}^{(k)}U_{mn}\alpha_m + \overline{Q}_{12}^{(k)}V_{mn}\beta_n - 2z\overline{Q}_{13}W_{mn}\alpha_m\beta_n\right)\cos\alpha_m x\cos\beta_n y\\ &+\left(zW_{mn}\left(\overline{Q}_{11}^{(k)}\alpha_m^2 + \overline{Q}_{12}^{(k)}\beta_n^2\right) - \overline{Q}_{13}^{(k)}\left(U_{mn}\beta_n + V_{mn}\alpha_m\right)\right)\sin\alpha_m x\sin\beta_n y\Big]\end{aligned}$$

$$\begin{aligned}\sigma_y^{(k)}(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[&\left(\overline{Q}_{12}^{(k)}U_{mn}\alpha_m + \overline{Q}_{22}^{(k)}V_{mn}\beta_n - 2z\overline{Q}_{23}W_{mn}\alpha_m\beta_n\right)\cos\alpha_m x\cos\beta_n y\\ &+\left(zW_{mn}\left(\overline{Q}_{12}^{(k)}\alpha_m^2 + \overline{Q}_{22}^{(k)}\beta_n^2\right) - \overline{Q}_{23}^{(k)}\left(U_{mn}\beta_n + V_{mn}\alpha_m\right)\right)\sin\alpha_m x\sin\beta_n y\Big]\end{aligned}$$

$$\begin{aligned}\tau_{xy}^{(k)}(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[&\left(\overline{Q}_{13}^{(k)}U_{mn}\alpha_m + \overline{Q}_{23}^{(k)}V_{mn}\beta_n - 2z\overline{Q}_{33}W_{mn}\alpha_m\beta_n\right)\cos\alpha_m x\cos\beta_n y\\ &+\left(zW_{mn}\left(\overline{Q}_{13}^{(k)}\alpha_m^2 + \overline{Q}_{23}^{(k)}\beta_n^2\right) - \overline{Q}_{33}^{(k)}\left(U_{mn}\beta_n + V_{mn}\alpha_m\right)\right)\sin\alpha_m x\sin\beta_n y\Big]\end{aligned} \tag{A-23}$$

Again according to CPT, the transverse shear stresses are zero.

First-Order Shear Deformation Theory Solution

The plate considered here is rectangular with hinged-edges (free in the tangential direction). In this case, the following boundary conditions apply along the edges:

$$
\begin{aligned}
&w(0,y)=w(a,y)=w(x,0)=w(x,b)=0\\
&u(0,y)=u(a,y)=v(x,0)=v(x,b)=0\\
&\psi_x(x,0)=\psi_x(x,b)=\psi_y(0,y)=\psi_y(a,y)=0\\
&M_x(0,y)=M_x(a,y)=M_y(x,0)=M_y(x,b)=0\\
&N_{xy}(0,y)=N_{xy}(a,y)=N_{xy}(x,0)=N_{xy}(x,b)=0
\end{aligned}
\tag{A-24}
$$

This solution requires five displacement functions. The following Navier series functions satisfy the FSDT governing plate equations in chapter III and the plate boundary conditions:

$$
\begin{aligned}
&u(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\\
&v(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
&w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
&\psi_x(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_{x_{mn}}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
&\psi_y(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_{y_{mn}}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}
\end{aligned}
\tag{A-25}
$$

Substituting these displacement fields into the governing FSDT equations 3-40 to 3-43 yields:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{12} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{13} & L_{23} & L_{33} & L_{34} & L_{35} \\ L_{14} & L_{24} & L_{34} & L_{44} & L_{45} \\ L_{15} & L_{25} & L_{35} & L_{45} & L_{55} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ \Psi x_{mn} \\ \Psi y_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ 0 \\ 0 \end{Bmatrix} \tag{A-26}$$

where the matrix terms are defined as:

$$\begin{aligned} L_{11} &= A_{11}\alpha_m^2 + A_{33}\beta_n^2 \\ L_{12} &= (A_{12} + A_{33})\alpha_m\beta_n \\ L_{13} &= 0 \\ L_{14} &= 2B_{13}\alpha_m\beta_n \\ L_{22} &= A_{33}\alpha_m^2 + A_{22}\beta_n^2 \\ L_{23} &= 0 \\ L_{33} &= A_{55}\alpha_m^2 + A_{44}\beta_n^2 \\ L_{34} &= A_{55}\alpha_m \\ L_{35} &= A_{44}\beta_n \\ L_{44} &= D_{11}\alpha_m^2 + D_{33}\beta_n^2 + A_{55} \\ L_{45} &= (D_{12} + D_{33})\alpha_m\beta_n \\ L_{55} &= D_{33}\alpha_m^2 + D_{22}\beta_n^2 + A_{44} \end{aligned} \tag{A-27}$$

and $\alpha_m = \dfrac{m\pi}{a}$, $\beta_n = \dfrac{n\pi}{b}$.

The solution of this equation yields the constants $(U_{mn}, V_{mn}, W_{mn}, \Psi x_{mn}, \Psi y_{mn})$ which can be substituted into equation A-25 to determine the deflection and rotations of any point on the plate mid-plane. The stresses may also be found by utilizing these terms are defined as follows:

$$\sigma_x^{(k)}(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\Big(\big(\overline{Q}_{11}^{(k)}U_{mn}\alpha_m+\overline{Q}_{12}^{(k)}V_{mn}\beta_n+z\overline{Q}_{13}^{(k)}\big(\Psi x_{mn}\beta_n+\Psi y_{mn}\alpha_m\big)\Big)\cos\alpha_m x$$
$$\cos\beta_n y-\Big(\overline{Q}_{13}^{(k)}\big(U_{mn}\beta_n+V_{mn}\alpha_m\big)+z\big(\overline{Q}_{11}^{(k)}\Psi x_{mn}\alpha_m+\overline{Q}_{12}^{(k)}\Psi y_{mn}\beta_n\big)\Big)$$
$$\sin\alpha_m x\sin\beta_n y\Big]$$

$$\sigma_y^{(k)}(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\Big(\big(\overline{Q}_{12}^{(k)}U_{mn}\alpha_m+\overline{Q}_{22}^{(k)}V_{mn}\beta_n+z\overline{Q}_{23}^{(k)}\big(\Psi x_{mn}\beta_n+\Psi y_{mn}\alpha_m\big)\Big)\cos\alpha_m x$$
$$\cos\beta_n y-\Big(\overline{Q}_{23}^{(k)}\big(U_{mn}\beta_n+V_{mn}\alpha_m\big)+z\big(\overline{Q}_{12}^{(k)}\Psi x_{mn}\alpha_m+\overline{Q}_{22}^{(k)}\Psi y_{mn}\beta_n\big)\Big)$$
$$\sin\alpha_m x\sin\beta_n y\Big]$$

$$\tau_{xy}^{(k)}(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\Big(\big(\overline{Q}_{13}^{(k)}U_{mn}\alpha_m+\overline{Q}_{23}^{(k)}V_{mn}\beta_n+z\overline{Q}_{33}^{(k)}\big(\Psi x_{mn}\beta_n+\Psi y_{mn}\alpha_m\big)\Big)\cos\alpha_m x$$
$$\cos\beta_n y-\Big(\overline{Q}_{33}^{(k)}\big(U_{mn}\beta_n+V_{mn}\alpha_m\big)+z\big(\overline{Q}_{13}^{(k)}\Psi x_{mn}\alpha_m+\overline{Q}_{23}^{(k)}\Psi y_{mn}\beta_n\big)\Big)$$
$$\sin\alpha_m x\sin\beta_n y\Big]$$

$$\tau_{yz}^{(k)}(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\overline{Q}_{44}^{(k)}\big(\Psi y_{mn}+W_{mn}\beta_n\big)\sin\alpha_m x\cos\beta_n y+\overline{Q}_{45}^{(k)}\big(\Psi x_{mn}+W_{mn}\alpha_n\big)$$
$$\cos\alpha_m x\sin\beta_n y\Big]$$

$$\tau_{xz}^{(k)}(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\overline{Q}_{45}^{(k)}\big(\Psi y_{mn}+W_{mn}\beta_n\big)\sin\alpha_m x\cos\beta_n y+\overline{Q}_{55}^{(k)}\big(\Psi x_{mn}+W_{mn}\alpha_n\big)$$
$$\cos\alpha_m x\sin\beta_n y\Big]$$

(A-28)

APPENDIX B

COMPUTER PROGRAM INFORMATION

COMPLATE Program Overview and Instructions

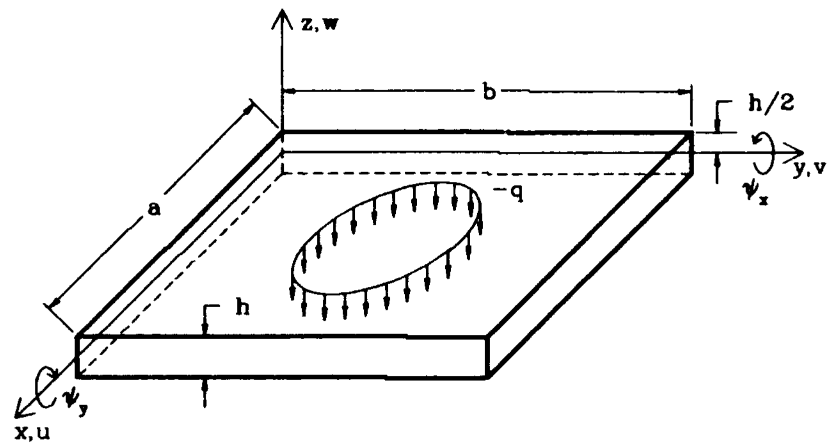

Figure 17: Definition of coordinates on a rectangular plate.

This program was developed to analyze general laminated composite plates of uniform thickness, h. The program uses first-order shear deformation of laminated orthotropic plates in the analysis. The results from the program include the laminate material property matrices, [A], [B], [D], the resulting mid-plane displacements and rotations (u, v, w, ψ_x, ψ_y) at plate nodal points, the laminate strains at the mid-plane and on the laminate surfaces, and the interlaminar stresses at lamina interfaces. The following paragraphs describe the user supplied requirements for the program. Figure 17 shows the coordinate system for an example rectangular plate.

1. Program Control Parameters

The program uses several parameters to control the scope and output of the program. TITLE is a one-line output heading used to describe the problem being analyzed. The next two parameters dictate the element type used in the program. IEL describes the order of the element (1 - linear, 2 - quadratic). NPE is the number of nodes per element (4 if IEL = 1, 8 or 9 if IEL = 2). IMESH is the parameter used to control whether a rectangular mesh is generated (IMESH = 1) or all mesh information is input for a general shape mesh (IMESH = 0). NPRNT is used to control the printing of element stiffness matrices and force vectors (0 - not printed, 1 - printed). ITEM indicates whether the analysis is static (= 0) or dynamic (= 1). The next three parameters, NTIME, NSTEP and NOZERO, are parameters for the dynamic analysis case and are describes under that section.

2. Shape of Plate (FEM Mesh)

This program has two methods for generating the plate mesh. Rectangular plate meshes can be generated by entering the number of divisions between nodal points along the x and y-axes, NX and NY, and the lengths of each division, DX(I) and DY(I) in the x and y directions respectively. These divisions do not have to have uniform lengths because each length is supplied.

General plate shapes require their meshes to be entered by the user. The user must provide the number of elements, NEM, the number of nodal points, NNM, the element connectivity of the nodal points, NOD(I,J), and the coordinates of the nodal points, X(I) and Y(I). The plate can therefore be any user defined shape including curved edges, but must have a uniform thickness. The user is encouraged to use other mesh generation programs to develop general shaped (non-rectangular) meshes.

3. Element Types

There are three types of quadrilateral isoparametric elements in this program. Each nodal point in the element has five degrees of freedom or generalized displacements (u, v, w, ψ_x, ψ_y). The three types of elements are four-node linear, eight-node quadratic, and nine-node quadratic elements as shown in Figure 18.

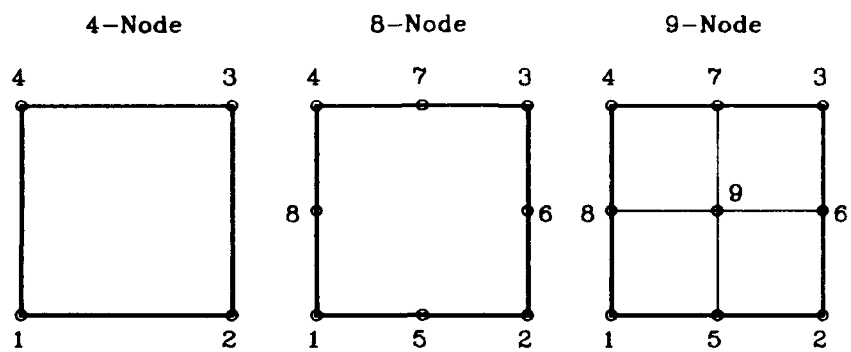

Figure 18: Element types and nodal point numbers.

The four, eight, and nine-noded elements produce element matrices of 20x20, 40x40, and 45x45, respectively. The order of these matrices is defined by the number of degrees of freedom per node. IEL and NPE control the type of element used as described in section 1.

The interpolation functions used for these elements are isoparametric and belong to the Lagrange family. This allows the element sides to be non-straight (for quadratic elements). Full-integration is used for all stiffness terms except for those terms involving traverse shear coefficients (A_{44}, A_{45}, A_{55}) in which a reduced-integration scheme is used. The reduced-integration is performed to prevent shear-locking effects.

4. Types of Materials and Stacking Sequence

This program was developed for general laminated composite plates including hybrid composites. The user supplies the number of different composite materials, NMTL in the laminate, and the material properties of each material (I):

Elastic moduli -	E_1 - E1(I), E_2 - E2(I)
Shear moduli -	G_{12} - G12(I), G_{13} - G13(I), G_{23} - G23(I)
Poisson's Ratio -	ν_{12} - ANU12(I)
Material density -	ρ - RHO(I)

Laminated plates have the following characteristics. The laminate is formed by stacking a number of orthotropic material layers, NLAY, in a desired sequence. Each layer, L, can have different principal material orientations defined in degrees from the x-axis, THETA(L), be made of different materials defined by the material number I, MTL(L), and have varied thicknesses, TH(L). The program is also able to analyze isotropic materials, orthotropic materials and hybrids made of these materials.

5. Displacement on Boundaries and Loading Conditions

This program accommodates general boundary conditions by allowing the user to specify the number of displacement conditions, NBDY, the location and direction of the displacement (u, v, w, ψ_x, ψ_y), IBDY(I), and the corresponding value of the generalized displacements, VBDY(I), for any nodal point.

There are two ways to apply loads to the plate. A uniformly distributed transverse load can be applied by specifying the magnitude and direction, P0 (positive-upward, negative-downward), of the pressure. Other loads are applied by specifying the number of specified forces, NSBF, the nodal point location and

generalized direction, IBSF(I), and magnitude of the load at the nodal points, VBSF(I), in the form (N_x, N_y, Q_z, M_x, M_y) corresponding to the generalized displacements.

By default, all generalized displacements are assumed to be free to move and all forces are assumed to be zero. The displacement and force vectors each have five times the number of nodes entries. The ordering of the displacement and force vectors are shown below:

$$\text{Displacement Vector - } \left\{\left\{u, v, w, \psi_x, \psi_y\right\}^{node1}, \left\{u, v, w, \psi_x, \psi_y\right\}^{node2}, \ldots\right\}$$

$$\text{Force Vector - } \left\{\left\{N_x, N_y, Q, M_x, M_y\right\}^{node1}, \left\{N_x, N_y, Q, M_x, M_y\right\}^{node2}, \ldots\right\} \qquad \text{(B-1)}$$

To change a boundary condition from the default conditions, the user must supply the position of the displacement or force in its respective vector and the value of that condition. For example, to specify a displacement of 10 in the y-direction on node 3, the user would give 12 for its position in the vector and 10.0 for its value.

One of each of the following boundary condition pairs must be specified along the plate edges :

$$N_n \text{ or } u_n, \quad N_{ns} \text{ or } u_s, \quad Q_n \text{ or } w, \quad M_n \text{ or } \psi_n, \quad M_{ns} \text{ or } \psi_s \qquad \text{(B-2)}$$

The following list describes the applicable force and displacement values for some commonly used boundary conditions. The coordinates n and s refer to the outward normal and the in-plane tangential axes along the plate edges respectively. For example on an an x-edge (x = a), n corresponds to the x-axis and s corresponds to the y-axis.

1. Simply-Supported Edge

$$N_n = 0,\ N_{ns} = 0,\ w = 0,\ M_n = 0,\ \psi_s = 0 \tag{B-3}$$

2. Hinged Edge - Free in the normal direction

$$N_n = 0,\ u_s = 0,\ w = 0,\ M_n = 0,\ \psi_s = 0 \tag{B-4}$$

3. Hinged Edge - Free in the tangential direction

$$u_n = 0,\ N_{ns} = 0,\ w = 0,\ M_n = 0,\ \psi_s = 0 \tag{B-5}$$

4. Clamped Edge

$$u_n = 0,\ u_s = 0,\ w = 0,\ \psi_n = 0,\ \psi_s = 0 \tag{B-6}$$

5. Free Edge

$$N_n = 0,\ N_{ns} = 0,\ Q_n = 0,\ M_n = 0,\ M_{ns} = 0 \tag{B-7}$$

6. Line of Symmetry (for symmetrical finite element problems)

$$u_n = 0,\ N_{ns} = 0,\ Q_n = 0,\ \psi_n = 0,\ M_{ns} = 0 \tag{B-8}$$

6. Dynamic Analysis

This program can analyze many different dynamic cases. For the general dynamic case (ITEM=1), the user must supply the number of time steps, NTIME, the time step size, DT, alpha from Newmark's method, ALFA, and the time step at which the transverse pressure is removed, NSTP. Additionally, initial generalized displacements and generalized velocities can be specified to add to the range of dynamic cases covered by the program. In this case, all the values of the initial displacements, GF0(I), and velocities, GF1(0) of each nodal point must be entered in the following order:

$$\text{Displacement Vector - } \left\{ \left\{ u, v, w, \psi_x, \psi_y \right\}^{\text{node1}}, \left\{ u, v, w, \psi_x, \psi_y \right\}^{\text{node2}}, \ldots \right\}$$

$$\text{Velocity Vector - } \left\{ \left\{ \dot{u}, \dot{v}, \dot{w}, \dot{\psi}_x, \dot{\psi}_y \right\}^{\text{node1}}, \left\{ \dot{u}, \dot{v}, \dot{w}, \dot{\psi}_x, \dot{\psi}_y \right\}^{\text{node2}}, \ldots \right\} \tag{B-9}$$

Computer Program Structure (COMPLATE)

Figure 19 shows the general program structure of COMPLATE with all subroutine calls. The dashed boxes denote optional features of the program depending on control parameters. This is provided as a reference for understanding the source code.

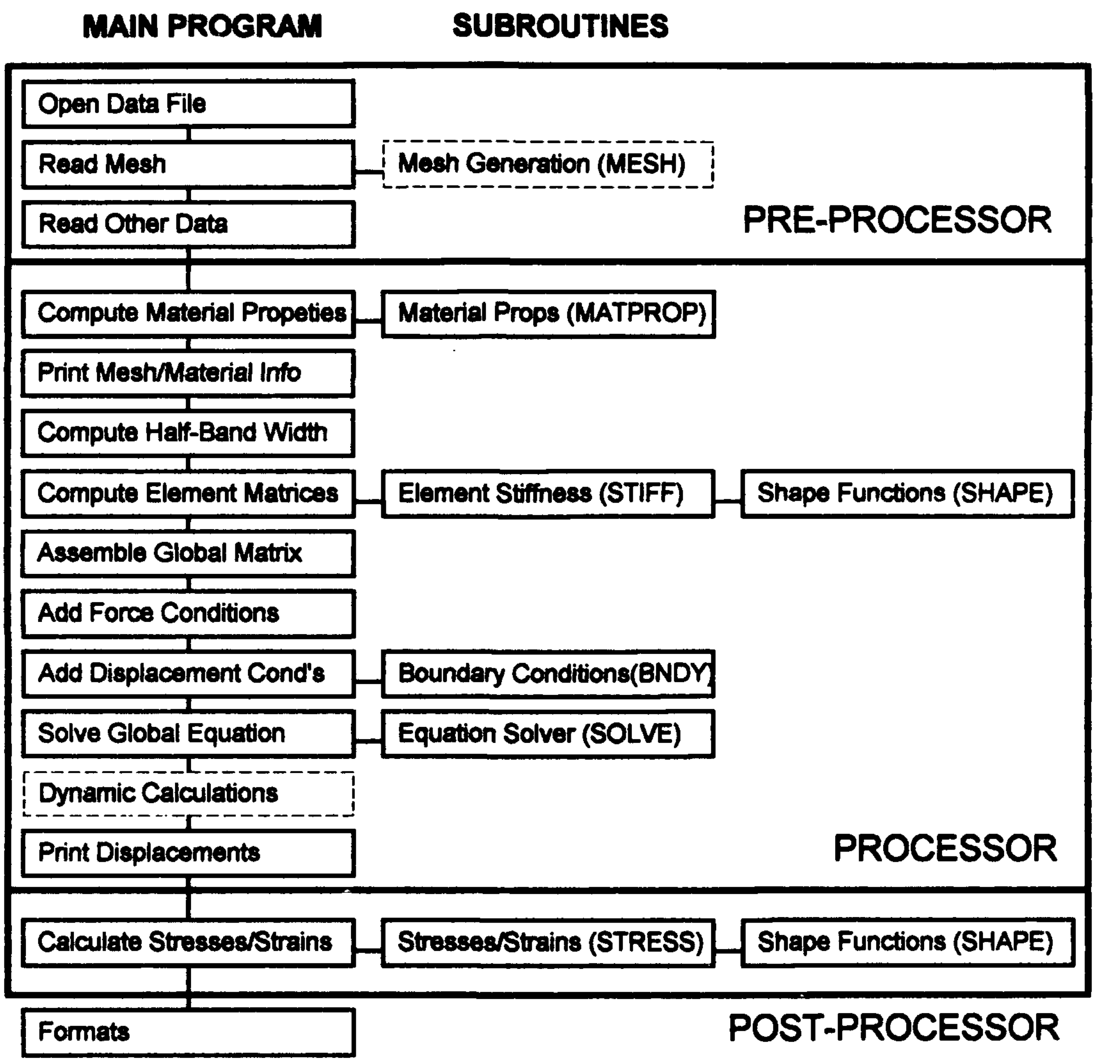

Figure 19: Computer program structure.

Data File Format

The following list describes the input format for COMPLATE. The data is read in using "free-form" format. This means the data may be separated by spaces or commas and the only important aspect is the order of variables in the data file. The following list provides a suggested format for ease in visually reading and adjusting the input file. Descriptions of all variables used in the program can be found at the top of the program source code or at the top of each subroutine. DATA LINE(S) outlines how the input is separated into sections or lines. Some DATA LINES require more than one actual input line depending on the number of variables to be read in. The information under the TYPE column describes the data type of the variable: Ann - character string of nn characters, I - integer (no decimal point); F - fixed-point (decimal point required). An example input file is given following this list for clarity.

VARIABLE	TYPE	VARIABLE DESCRIPTION AND NOTES
DATA LINE 1		OUTPUT HEADING - (80 characters)
TITLE	A80	Title for output file - description of problem
DATA LINE 2		PROGRAM PARAMETERS
IEL	I	Element type (1 = four node, 2 = eight or nine node)
NPE	I	Nodes per element (4 if IEL=1, 8 or 9 if IEL=2)
IMESH	I	Indicator for rectangular mesh generation (0 - all element information is read in 1 - rectangular mesh is generated)
NPRNT	I	Indicator for printing the element stiffness matrices and force vectors (0 - no printing, 1 - printing)
ITEM	I	Indicator for transient analysis (0 - static, 1 - transient)
NTIME	I	Total number of time steps (0 for ITEM = 0)
NSTP	I	Time step number at which loading is removed
NOZERO	I	Indicator for initial transient load conditions (0 - zero initial load, 1 - non-zero initial load)

*** SKIP LINES 3, 4, AND 5 IF IMESH = 1 (MESH GENERATED) ***

<u>DATA LINE 3</u>		READ MESH PARAMETERS
NEM	I	Number of elements in the mesh
NNM	I	Number of nodes in the mesh
<u>DATA LINE(S) 4</u>		MESH CONNECTIVITY - NEM lines, NPE per line
NOD(I,J)	F	Connectivity of I-th element (J = 1,NPE)
<u>DATA LINE(S) 5</u>		NODAL COORDINATES
X(I), Y(I)	F	Global coordinates of I-th node

*** SKIP LINES 6, 7, AND 8 IF IMESH = 0 (GENERAL MESH) ***

<u>DATA LINE 6</u>		MESH GENERATION PARAMETERS
NX	I	Number of element subdivisions along the x-axis
NY	I	Number of element subdivisions along the y-axis
<u>DATA LINE(S) 7</u>		X-DIVISIONS - IEL*NX Entries
DX(I)	F	Distance between two nodes along the x-axis
<u>DATA LINE(S) 8</u>		Y-DIVISIONS - IEL*NY Entries
DY(I)	F	Distance between two nodes along the y-axis
<u>DATA LINE 9</u>		MATERIALS
NMTL	I	Number of different materials in the laminate
<u>DATA LINE(S) 10</u>		MATERIAL I PROPERTIES - NMTL lines
E1(I)	F	Modulus along fiber direction (1-direction)
E2(I)	F	Modulus transverse to fiber direction (2-direction)
G12(I)	F	In-plane shear modulus oriented along fiber direction
G13(I)	F	Shear modulus with respect to 1-3 plane
G23(I)	F	Shear modulus with respect to 2-3 plane
ANU12(I)	F	In-plane Poisson's ratio (1-2 plane)
RHO(I)	F	Material Density

<u>DATA LINE 11</u>		NUMBER OF LAMINA
NLAY	I	Number of lamina (layers) in the laminate
<u>DATA LINE(S) 12</u>		LAMINA PROPERTIES - NLAY lines
MTL(I)	I	Material number of I-th lamina
THETA(I)	F	Fiber orientation angle of I-th lamina (in degrees)
TH(I)	F	Thickness of I-th lamina
<u>DATA LINE 13</u>		UNIFORM TRANSVERSE LOADING
P0	F	Intensity of uniformly distributed transverse load
<u>DATA LINE 14</u>		SPECIFIED DISPLACEMENTS (cannot be zero)
NBDY	I	Number of specified generalized displacements
<u>DATA LINE(S) 15</u>		DISPLACEMENTS - NBDY entries
IBDY(I)	I	Location/direction of specified displacement I Order - by node number and (u, v, w, ψ_x, ψ_y)
<u>DATA LINE(S) 16</u>		DISPLACEMENT VALUES - NBDY entries
VBDY(I)	F	Value of displacement corresponding to IBDY(I)
<u>DATA LINE 17</u>		SPECIFIED FORCES
NBSF	I	Number of specified generalized forces

*** SKIP LINES 18 AND 19 IF NBSF = 0 (NO FORCE CONDITIONS OTHER THAN TRANSVERSE PRESSURE) ***

<u>DATA LINE(S) 18</u>		FORCES - NBSF entries
IBSF(I)	I	Location/direction of specified generalized forces Order - by node number and (N_x, N_y, Q_z, M_x, M_y)
<u>DATA LINE(S) 19</u>		FORCE VALUES - NBSF entries, 8 per line
VBSF(I)	F	Value of specified force corresponding to IBSF(I)

*** SKIP LINES 20, 21, AND 22 IF ITEM = 0 (STATIC ANALYSIS) ***

DATA LINE 20		TRANSIENT PARAMETERS
DT	F	Time step for transient analysis
ALFA	F	Parameter in Newmark's method

*** SKIP LINES 21 AND 22 IF NOZERO = 0 (NO INITIAL CONDITIONS) ***

DATA LINE(S) 21		INITIAL DISPLACEMENTS. - NNM*NDF entries
GF0(I)	F	Initial value of generalized displacement for I-th degree of freedom (DOF)
DATA LINE(S) 22		INITIAL VELOCITY - NNM*NDF entries
GF1(I)	F	Initial value of generalized velocity for I-th DOF

Sample Input Data File

```
STATIC BENDING OF A SIMPLY SUPPORTED PLATE UNDER UNIFORM LOAD (2X2Q9 MESH)
   2    9    1    0    0    1    0    0
   2    2
 1.25      1.25      1.25      1.25
 1.25      1.25      1.25      1.25
 1
 25.E6     1.E6     0.5E6     0.5E6     0.2E6     0.25      0.3
 3
 1          0.0      0.033333
 1         90.0      0.033334
 1          0.0      0.033333
 1.0
  37
   1    2    4    5    7   10   12   15   17   20   22   23   25   26   29
  48   50   51   54   73   75   76   79   98  100  101  103  104  108  109
 113  114  118  119  123  124  125
 0.0       0.0       0.0       0.0       0.0       0.0       0.0       0.0
 0.0       0.0       0.0       0.0       0.0       0.0       0.0       0.0
 0.0       0.0       0.0       0.0       0.0       0.0       0.0       0.0
 0.0       0.0       0.0       0.0       0.0       0.0       0.0       0.0
 0.0       0.0       0.0       0.0       0.0
   0
```

Sample Output File

The following output corresponds to the previous input file.

```
STATIC BENDING OF A SIMPLY SUPPORTED PLATE UNDER UNIFORM LOAD (2X2Q9 MESH)

    ELEMENT TYPE(1=LINEAR,2=QUADRATIC) = 2      NODES PER ELEMENT= 9
         ACTUAL NUMBER OF ELEMENTS IN THE MESH=  4
         NUMBER OF NODES IN THE MESH = 25
         DEGREES OF FREEDOM = 5

    MATERIAL  1 PROPERTIES:
         MODULUS,E1= 0.25000E+08
         MODULUS,E2= 0.10000E+07
         SHEAR MODULUS,G12= 0.50000E+06
         SHEAR MODULUS,G13= 0.50000E+06
         SHEAR MODULUS,G23= 0.20000E+06
         POISSONS RATIO,NU12= 0.25000E+00
         MATERIAL DENSITY,RHO= 0.30000E+00


    LAMINATE STACKING SEQUENCE

       LAYER    MTL #     THETA      THICKNESS
         1        1      0.00000    0.33333E-01
         2        1     90.00000    0.33334E-01
         3        1      0.00000    0.33333E-01

       TOTAL THICKNESS = 0.10000E+00

    LAMINATE PLATE PROPERTIES

       A MATRIX TERMS
          0.17042E+07    0.25063E+05    0.00000E+00
          0.25063E+05    0.90227E+06    0.00000E+00
          0.00000E+00    0.00000E+00    0.50000E+05

       B MATRIX TERMS
          0.00000E+00    0.00000E+00    0.00000E+00
          0.00000E+00    0.00000E+00    0.00000E+00
          0.00000E+00    0.00000E+00    0.00000E+00

       D MATRIX TERMS
          0.20143E+04    0.20886E+02    0.00000E+00
          0.20886E+02    0.15781E+03    0.00000E+00
          0.00000E+00    0.00000E+00    0.41667E+02

       SHEAR TERMS: A44, A45, A55
          0.25000E+05    0.00000E+00    0.33333E+05

       INERTIAL TERMS I1, I2, I3
          0.30000E-01    0.00000E+00    0.25000E-04

    NUMBER OF SPECIFIED DISPLACEMENTS=   37
```

SPECIFED DISPLACEMENTS AND THEIR VALUES FOLLOW:

1	0.00000E+00	2	0.00000E+00	4	0.00000E+00
5	0.00000E+00	7	0.00000E+00	10	0.00000E+00
12	0.00000E+00	15	0.00000E+00	17	0.00000E+00
20	0.00000E+00	22	0.00000E+00	23	0.00000E+00
25	0.00000E+00	26	0.00000E+00	29	0.00000E+00
48	0.00000E+00	50	0.00000E+00	51	0.00000E+00
54	0.00000E+00	73	0.00000E+00	75	0.00000E+00
76	0.00000E+00	79	0.00000E+00	98	0.00000E+00
100	0.00000E+00	101	0.00000E+00	103	0.00000E+00
104	0.00000E+00	108	0.00000E+00	109	0.00000E+00
113	0.00000E+00	114	0.00000E+00	118	0.00000E+00
119	0.00000E+00	123	0.00000E+00	124	0.00000E+00
125	0.00000E+00				

UNIFORMLY DISTRIBUTED LOAD, PO = 0.10000E+01

NUMBER OF SPECIFIED FORCES= 0
SPECIFIED FORCE DEGREES OF FREEDOM AND THEIR VALUES FOLLOW:

BOOLEAN (CONNECTIVITY) MATRIX-NOD(I,J)

1	1	3	13	11	2	8	12	6	7
2	3	5	15	13	4	10	14	8	9
3	11	13	23	21	12	18	22	16	17
4	13	15	25	23	14	20	24	18	19

COORDINATES OF THE GLOBAL NODES:

1	0.00000E+00	0.00000E+00	2	0.12500E+01	0.00000E+00
3	0.25000E+01	0.00000E+00	4	0.37500E+01	0.00000E+00
5	0.50000E+01	0.00000E+00	6	0.00000E+00	0.12500E+01
7	0.12500E+01	0.12500E+01	8	0.25000E+01	0.12500E+01
9	0.37500E+01	0.12500E+01	10	0.50000E+01	0.12500E+01
11	0.00000E+00	0.25000E+01	12	0.12500E+01	0.25000E+01
13	0.25000E+01	0.25000E+01	14	0.37500E+01	0.25000E+01
15	0.50000E+01	0.25000E+01	16	0.00000E+00	0.37500E+01
17	0.12500E+01	0.37500E+01	18	0.25000E+01	0.37500E+01
19	0.37500E+01	0.37500E+01	20	0.50000E+01	0.37500E+01
21	0.00000E+00	0.50000E+01	22	0.12500E+01	0.50000E+01
23	0.25000E+01	0.50000E+01	24	0.37500E+01	0.50000E+01
25	0.50000E+01	0.50000E+01			

HALF BAND WIDTH OF GLOBAL STIFFNESS MATRIX = 65

GENERALIZED DISPLACEMENTS (U,V,W,SX,SY) PER NODE

1	0.00000E+00	0.00000E+00	0.67040E-01	0.00000E+00	0.00000E+00
2	0.00000E+00	0.00000E+00	0.62012E-01	0.78426E-02	0.00000E+00
3	0.00000E+00	0.00000E+00	0.47771E-01	0.14677E-01	0.00000E+00
4	0.00000E+00	0.00000E+00	0.25989E-01	0.19509E-01	0.00000E+00
5	0.00000E+00	0.00000E+00	0.00000E+00	0.21343E-01	0.00000E+00
6	0.00000E+00	0.00000E+00	0.63455E-01	0.00000E+00	0.59374E-02
7	0.00000E+00	0.00000E+00	0.58703E-01	0.74048E-02	0.54801E-02
8	0.00000E+00	0.00000E+00	0.45230E-01	0.13868E-01	0.41977E-02
9	0.00000E+00	0.00000E+00	0.24611E-01	0.18440E-01	0.22724E-02

10	0.00000E+00	0.00000E+00	0.00000E+00	0.20175E-01	0.00000E+00
11	0.00000E+00	0.00000E+00	0.51706E-01	0.00000E+00	0.13060E-01
12	0.00000E+00	0.00000E+00	0.47869E-01	0.59961E-02	0.12039E-01
13	0.00000E+00	0.00000E+00	0.36951E-01	0.11279E-01	0.91828E-02
14	0.00000E+00	0.00000E+00	0.20143E-01	0.15090E-01	0.49446E-02
15	0.00000E+00	0.00000E+00	0.00000E+00	0.16560E-01	0.00000E+00
16	0.00000E+00	0.00000E+00	0.29794E-01	0.00000E+00	0.21322E-01
17	0.00000E+00	0.00000E+00	0.27620E-01	0.34017E-02	0.19733E-01
18	0.00000E+00	0.00000E+00	0.21398E-01	0.64412E-02	0.15228E-01
19	0.00000E+00	0.00000E+00	0.11718E-01	0.87228E-02	0.82896E-02
20	0.00000E+00	0.00000E+00	0.00000E+00	0.96390E-02	0.00000E+00
21	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.25620E-01
22	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.23769E-01
23	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.18493E-01
24	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.10180E-01
25	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00

LAMINATE STRAINS AND STRESSES AT GAUSS POINTS

(X-COORD,Y-COORD)

LOC	EPSILONX	EPSILONY	GAMMAXY	KAPPAX	KAPPAY	KAPPAXY
LAY	Z-COORD	SIGMAX	SIGMAY	SIGMAXY	SIGMAYZ	SIGMAXZ
(0.5283E+00, 0.5283E+00)						
MID	0.0000E+00	0.0000E+00	0.0000E+00	0.6283E-02	0.4604E-02	-0.2520E-03
BOT	-0.3141E-03	-0.2302E-03	0.1260E-04			
TOP	0.3141E-03	0.2302E-03	-0.1260E-04			
1	-0.5000E-01	-0.7931E+04	-0.3095E+03	0.6300E+01	0.4125E+00	-0.8524E+01
1	-0.1667E-01	-0.2644E+04	-0.1032E+03	0.2100E+01	0.4125E+00	-0.8524E+01
2	-0.1667E-01	-0.1242E+03	-0.1949E+04	0.2100E+01	0.1031E+01	-0.3410E+01
2	0.1667E-01	0.1242E+03	0.1949E+04	-0.2100E+01	0.1031E+01	-0.3410E+01
3	0.1667E-01	0.2644E+04	0.1032E+03	-0.2100E+01	0.4125E+00	-0.8524E+01
3	0.5000E-01	0.7931E+04	0.3095E+03	-0.6300E+01	0.4125E+00	-0.8524E+01
(0.5283E+00, 0.1972E+01)						
MID	0.0000E+00	0.0000E+00	0.0000E+00	0.5419E-02	0.5678E-02	-0.1055E-02
BOT	-0.2710E-03	-0.2839E-03	0.5277E-04			
TOP	0.2710E-03	0.2839E-03	-0.5277E-04			
1	-0.5000E-01	-0.6862E+04	-0.3525E+03	0.2639E+02	0.4592E+00	-0.7366E+01
1	-0.1667E-01	-0.2287E+04	-0.1175E+03	0.8795E+01	0.4592E+00	-0.7366E+01
2	-0.1667E-01	-0.1143E+03	-0.2394E+04	0.8795E+01	0.1148E+01	-0.2946E+01
2	0.1667E-01	0.1143E+03	0.2394E+04	-0.8795E+01	0.1148E+01	-0.2946E+01
3	0.1667E-01	0.2287E+04	0.1175E+03	-0.8795E+01	0.4592E+00	-0.7366E+01
3	0.5000E-01	0.6862E+04	0.3525E+03	-0.2639E+02	0.4592E+00	-0.7366E+01
(0.1972E+01, 0.5283E+00)						
MID	0.0000E+00	0.0000E+00	0.0000E+00	0.5359E-02	0.3815E-02	-0.8633E-03
BOT	-0.2680E-03	-0.1907E-03	0.4316E-04			
TOP	0.2680E-03	0.1907E-03	-0.4316E-04			
1	-0.5000E-01	-0.6763E+04	-0.2584E+03	0.2158E+02	0.3032E+00	-0.3156E+02
1	-0.1667E-01	-0.2255E+04	-0.8612E+02	0.7194E+01	0.3032E+00	-0.3156E+02
2	-0.1667E-01	-0.1055E+03	-0.1616E+04	0.7194E+01	0.7579E+00	-0.1263E+02
2	0.1667E-01	0.1055E+03	0.1616E+04	-0.7194E+01	0.7579E+00	-0.1263E+02
3	0.1667E-01	0.2255E+04	0.8612E+02	-0.7194E+01	0.3032E+00	-0.3156E+02
3	0.5000E-01	0.6763E+04	0.2584E+03	-0.2158E+02	0.3032E+00	-0.3156E+02

```
( 0.1972E+01, 0.1972E+01)
 MID    0.0000E+00  0.0000E+00  0.0000E+00  0.4653E-02  0.4676E-02 -0.3595E-02
 BOT   -0.2327E-03 -0.2338E-03  0.1797E-03
 TOP    0.2327E-03  0.2338E-03 -0.1797E-03
  1  -0.5000E-01  -0.5890E+04 -0.2927E+03  0.8987E+02  0.4438E+00 -0.2798E+02
  1  -0.1667E-01  -0.1963E+04 -0.9758E+02  0.2996E+02  0.4438E+00 -0.2798E+02
  2  -0.1667E-01  -0.9728E+02 -0.1973E+04  0.2996E+02  0.1110E+01 -0.1119E+02
  2   0.1667E-01   0.9728E+02  0.1973E+04 -0.2996E+02  0.1110E+01 -0.1119E+02
  3   0.1667E-01   0.1963E+04  0.9758E+02 -0.2996E+02  0.4438E+00 -0.2798E+02
  3   0.5000E-01   0.5890E+04  0.2927E+03 -0.8987E+02  0.4438E+00 -0.2798E+02

( 0.3028E+01, 0.5283E+00)
 MID    0.0000E+00  0.0000E+00  0.0000E+00  0.4011E-02  0.2702E-02 -0.1229E-02
 BOT   -0.2005E-03 -0.1351E-03  0.6144E-04
 TOP    0.2005E-03  0.1351E-03 -0.6144E-04
  1  -0.5000E-01  -0.5060E+04 -0.1857E+03  0.3072E+02  0.1910E+00 -0.4811E+02
  1  -0.1667E-01  -0.1687E+04 -0.6191E+02  0.1024E+02  0.1910E+00 -0.4811E+02
  2  -0.1667E-01  -0.7830E+02 -0.1146E+04  0.1024E+02  0.4775E+00 -0.1924E+02
  2   0.1667E-01   0.7830E+02  0.1146E+04 -0.1024E+02  0.4775E+00 -0.1924E+02
  3   0.1667E-01   0.1687E+04  0.6191E+02 -0.1024E+02  0.1910E+00 -0.4811E+02
  3   0.5000E-01   0.5060E+04  0.1857E+03 -0.3072E+02  0.1910E+00 -0.4811E+02

( 0.3028E+01, 0.1972E+01)
 MID    0.0000E+00  0.0000E+00  0.0000E+00  0.3516E-02  0.3280E-02 -0.5039E-02
 BOT   -0.1758E-03 -0.1640E-03  0.2520E-03
 TOP    0.1758E-03  0.1640E-03 -0.2520E-03
  1  -0.5000E-01  -0.4447E+04 -0.2085E+03  0.1260E+03  0.1813E+00 -0.4337E+02
  1  -0.1667E-01  -0.1482E+04 -0.6949E+02  0.4200E+02  0.1813E+00 -0.4337E+02
  2  -0.1667E-01  -0.7244E+02 -0.1385E+04  0.4200E+02  0.4533E+00 -0.1735E+02
  2   0.1667E-01   0.7244E+02  0.1385E+04 -0.4200E+02  0.4533E+00 -0.1735E+02
  3   0.1667E-01   0.1482E+04  0.6949E+02 -0.4200E+02  0.1813E+00 -0.4337E+02
  3   0.5000E-01   0.4447E+04  0.2085E+03 -0.1260E+03  0.1813E+00 -0.4337E+02

( 0.4472E+01, 0.5283E+00)
 MID    0.0000E+00  0.0000E+00  0.0000E+00  0.1266E-02  0.7917E-03 -0.1505E-02
 BOT   -0.6329E-04 -0.3958E-04  0.7524E-04
 TOP    0.6329E-04  0.3958E-04 -0.7524E-04
  1  -0.5000E-01  -0.1596E+04 -0.5554E+02  0.3762E+02  0.1455E+00 -0.7036E+02
  1  -0.1667E-01  -0.5321E+03 -0.1852E+02  0.1254E+02  0.1455E+00 -0.7036E+02
  2  -0.1667E-01  -0.2446E+02 -0.3360E+03  0.1254E+02  0.3639E+00 -0.2814E+02
  2   0.1667E-01   0.2446E+02  0.3360E+03 -0.1254E+02  0.3639E+00 -0.2814E+02
  3   0.1667E-01   0.5321E+03  0.1852E+02 -0.1254E+02  0.1455E+00 -0.7036E+02
  3   0.5000E-01   0.1596E+04  0.5554E+02 -0.3762E+02  0.1455E+00 -0.7036E+02

( 0.4472E+01, 0.1972E+01)
 MID    0.0000E+00  0.0000E+00  0.0000E+00  0.1121E-02  0.9492E-03 -0.6048E-02
 BOT   -0.5606E-04 -0.4746E-04  0.3024E-03
 TOP    0.5606E-04  0.4746E-04 -0.3024E-03
  1  -0.5000E-01  -0.1417E+04 -0.6163E+02  0.1512E+03  0.2988E-01 -0.6418E+02
  1  -0.1667E-01  -0.4723E+03 -0.2054E+02  0.5040E+02  0.2988E-01 -0.6418E+02
  2  -0.1667E-01  -0.2270E+02 -0.4012E+03  0.5040E+02  0.7469E-01 -0.2567E+02
  2   0.1667E-01   0.2270E+02  0.4012E+03 -0.5040E+02  0.7469E-01 -0.2567E+02
  3   0.1667E-01   0.4723E+03  0.2054E+02 -0.5040E+02  0.2988E-01 -0.6418E+02
  3   0.5000E-01   0.1417E+04  0.6163E+02 -0.1512E+03  0.2988E-01 -0.6418E+02

( 0.5283E+00, 0.3028E+01)
 MID    0.0000E+00  0.0000E+00  0.0000E+00  0.4034E-02  0.6765E-02 -0.1782E-02
 BOT   -0.2017E-03 -0.3382E-03  0.8912E-04
```

```
 TOP     0.2017E-03  0.3382E-03 -0.8912E-04
  1  -0.5000E-01   -0.5139E+04 -0.3896E+03  0.4456E+02 -0.1323E+01 -0.4661E+01
  1  -0.1667E-01   -0.1713E+04 -0.1299E+03  0.1485E+02 -0.1323E+01 -0.4661E+01
  2  -0.1667E-01   -0.9565E+02 -0.2843E+04  0.1485E+02 -0.3306E+01 -0.1865E+01
  2   0.1667E-01    0.9565E+02  0.2843E+04 -0.1485E+02 -0.3306E+01 -0.1865E+01
  3   0.1667E-01    0.1713E+04  0.1299E+03 -0.1485E+02 -0.1323E+01 -0.4661E+01
  3   0.5000E-01    0.5139E+04  0.3896E+03 -0.4456E+02 -0.1323E+01 -0.4661E+01

(  0.5283E+00,  0.4472E+01)
 MID     0.0000E+00  0.0000E+00  0.0000E+00  0.1239E-02  0.3158E-02 -0.2425E-02
 BOT    -0.6193E-04 -0.1579E-03  0.1213E-03
 TOP     0.6193E-04  0.1579E-03 -0.1213E-03
  1  -0.5000E-01   -0.1592E+04 -0.1738E+03  0.6063E+02 -0.8194E+01 -0.1608E+01
  1  -0.1667E-01   -0.5306E+03 -0.5794E+02  0.2021E+02 -0.8194E+01 -0.1608E+01
  2  -0.1667E-01   -0.3389E+02 -0.1324E+04  0.2021E+02 -0.2049E+02 -0.6432E+00
  2   0.1667E-01    0.3389E+02  0.1324E+04 -0.2021E+02 -0.2049E+02 -0.6432E+00
  3   0.1667E-01    0.5306E+03  0.5794E+02 -0.2021E+02 -0.8194E+01 -0.1608E+01
  3   0.5000E-01    0.1592E+04  0.1738E+03 -0.6063E+02 -0.8194E+01 -0.1608E+01

(  0.1972E+01,  0.3028E+01)
 MID     0.0000E+00  0.0000E+00  0.0000E+00  0.3510E-02  0.5699E-02 -0.6142E-02
 BOT    -0.1755E-03 -0.2849E-03  0.3071E-03
 TOP     0.1755E-03  0.2849E-03 -0.3071E-03
  1  -0.5000E-01   -0.4470E+04 -0.3297E+03  0.1535E+03 -0.9461E+00 -0.1815E+02
  1  -0.1667E-01   -0.1490E+04 -0.1099E+03  0.5118E+02 -0.9461E+00 -0.1815E+02
  2  -0.1667E-01   -0.8246E+02 -0.2395E+04  0.5118E+02 -0.2365E+01 -0.7258E+01
  2   0.1667E-01    0.8246E+02  0.2395E+04 -0.5118E+02 -0.2365E+01 -0.7258E+01
  3   0.1667E-01    0.1490E+04  0.1099E+03 -0.5118E+02 -0.9461E+00 -0.1815E+02
  3   0.5000E-01    0.4470E+04  0.3297E+03 -0.1535E+03 -0.9461E+00 -0.1815E+02

(  0.1972E+01,  0.4472E+01)
 MID     0.0000E+00  0.0000E+00  0.0000E+00  0.1096E-02  0.2723E-02 -0.8488E-02
 BOT    -0.5479E-04 -0.1361E-03  0.4244E-03
 TOP     0.5479E-04  0.1361E-03 -0.4244E-03
  1  -0.5000E-01   -0.1407E+04 -0.1502E+03  0.2122E+03 -0.7229E+01 -0.6760E+01
  1  -0.1667E-01   -0.4691E+03 -0.5007E+02  0.7074E+02 -0.7229E+01 -0.6760E+01
  2  -0.1667E-01   -0.2968E+02 -0.1142E+04  0.7074E+02 -0.1807E+02 -0.2704E+01
  2   0.1667E-01    0.2968E+02  0.1142E+04 -0.7074E+02 -0.1807E+02 -0.2704E+01
  3   0.1667E-01    0.4691E+03  0.5007E+02 -0.7074E+02 -0.7229E+01 -0.6760E+01
  3   0.5000E-01    0.1407E+04  0.1502E+03 -0.2122E+03 -0.7229E+01 -0.6760E+01

(  0.3028E+01,  0.3028E+01)
 MID     0.0000E+00  0.0000E+00  0.0000E+00  0.2727E-02  0.4125E-02 -0.8732E-02
 BOT    -0.1364E-03 -0.2062E-03  0.4366E-03
 TOP     0.1364E-03  0.2062E-03 -0.4366E-03
  1  -0.5000E-01   -0.3470E+04 -0.2409E+03  0.2183E+03 -0.3552E+00 -0.2968E+02
  1  -0.1667E-01   -0.1157E+04 -0.8031E+02  0.7277E+02 -0.3552E+00 -0.2968E+02
  2  -0.1667E-01   -0.6280E+02 -0.1734E+04  0.7277E+02 -0.8881E+00 -0.1187E+02
  2   0.1667E-01    0.6280E+02  0.1734E+04 -0.7277E+02 -0.8881E+00 -0.1187E+02
  3   0.1667E-01    0.1157E+04  0.8031E+02 -0.7277E+02 -0.3552E+00 -0.2968E+02
  3   0.5000E-01    0.3470E+04  0.2409E+03 -0.2183E+03 -0.3552E+00 -0.2968E+02

(  0.3028E+01,  0.4472E+01)
 MID     0.0000E+00  0.0000E+00  0.0000E+00  0.8835E-03  0.2059E-02 -0.1239E-01
 BOT    -0.4418E-04 -0.1029E-03  0.6196E-03
 TOP     0.4418E-04  0.1029E-03 -0.6196E-03
  1  -0.5000E-01   -0.1133E+04 -0.1143E+03  0.3098E+03 -0.5192E+01 -0.1112E+02
  1  -0.1667E-01   -0.3777E+03 -0.3809E+02  0.1033E+03 -0.5192E+01 -0.1112E+02
```

```
  2  -0.1667E-01  -0.2336E+02 -0.8637E+03  0.1033E+03 -0.1298E+02 -0.4450E+01
  2   0.1667E-01   0.2336E+02  0.8637E+03 -0.1033E+03 -0.1298E+02 -0.4450E+01
  3   0.1667E-01   0.3777E+03  0.3809E+02 -0.1033E+03 -0.5192E+01 -0.1112E+02
  3   0.5000E-01   0.1133E+04  0.1143E+03 -0.3098E+03 -0.5192E+01 -0.1112E+02

(  0.4472E+01,  0.3028E+01)
 MID    0.0000E+00  0.0000E+00  0.0000E+00  0.9018E-03  0.1233E-02 -0.1057E-01
 BOT   -0.4509E-04 -0.6165E-04  0.5287E-03
 TOP    0.4509E-04  0.6165E-04 -0.5287E-03
  1  -0.5000E-01  -0.1146E+04 -0.7310E+02  0.2643E+03  0.7850E-01 -0.4947E+02
  1  -0.1667E-01  -0.3819E+03 -0.2437E+02  0.8811E+02  0.7850E-01 -0.4947E+02
  2  -0.1667E-01  -0.2022E+02 -0.5188E+03  0.8811E+02  0.1963E+00 -0.1979E+02
  2   0.1667E-01   0.2022E+02  0.5188E+03 -0.8811E+02  0.1963E+00 -0.1979E+02
  3   0.1667E-01   0.3819E+03  0.2437E+02 -0.8811E+02  0.7850E-01 -0.4947E+02
  3   0.5000E-01   0.1146E+04  0.7310E+02 -0.2643E+03  0.7850E-01 -0.4947E+02

(  0.4472E+01,  0.4472E+01)
 MID    0.0000E+00  0.0000E+00  0.0000E+00  0.3065E-03  0.6503E-03 -0.1545E-01
 BOT   -0.1533E-04 -0.3251E-04  0.7723E-03
 TOP    0.1533E-04  0.3251E-04 -0.7723E-03
  1  -0.5000E-01  -0.3923E+03 -0.3644E+02  0.3861E+03 -0.2025E+01 -0.2162E+02
  1  -0.1667E-01  -0.1308E+03 -0.1215E+02  0.1287E+03 -0.2025E+01 -0.2162E+02
  2  -0.1667E-01  -0.7838E+01 -0.2729E+03  0.1287E+03 -0.5063E+01 -0.8647E+01
  2   0.1667E-01   0.7838E+01  0.2729E+03 -0.1287E+03 -0.5063E+01 -0.8647E+01
  3   0.1667E-01   0.1308E+03  0.1215E+02 -0.1287E+03 -0.2025E+01 -0.2162E+02
  3   0.5000E-01   0.3923E+03  0.3644E+02 -0.3861E+03 -0.2025E+01 -0.2162E+02
```

APPENDIX C

COMPLATE - COMPUTER PROGRAM SOURCE CODE

```
C          C O M P U T E R   P R O G R A M   C O M P L A T E
C         (STATIC AND TRANSIENT ANALYSIS OF COMPOSITE PLATES)
C
C     Revision of program PLATE by J.N. Reddy for orthotropic plates
C       Source: Reddy, J.N. "Introduction to the Finite Element Method",
C             McGraw-Hill, New York (1984).
C
C     Revision by: Brett A. Pauer, The Ohio State University
C
C ...................................................................
C .                                                                 .
C .     D E S C R I P T I O N   O F   T H E   V A R I A B L E S     .
C .                                                                 .
C .  A(I,J)......EXTENSIONAL STIFFNESS MATRIX (I,J=1,2,3)           .
C .  A(K,L)......SHEAR TERMS OF STIFFNESS MATRIX (K,L=4,5)          .
C .  A0,A1,A2,A3,A4...PARAMETERS IN THE TIME-APPROXIMATION SCHEME   .
C .  AK..........SHEAR CORRECTION COEFFICIENT                       .
C .  ALFA........PARAMETER IN THE NEWMARK SCHEME                    .
C .  ANU12(I)....POISSON'S RATIO OF MATERIAL                        .
C .  B(I,J)......COUPLING STIFFNESS MATRIX (I,J=1,2,3)              .
C .  BETA........PARAMETER IN THE NEWMARK SCHEME                    .
C .  D(I,J)......BENDING STIFFNESS MATRIX (I,J=1,2,3)               .
C .  DX(I),DY(I).DISTANCE BETWEEN NODES IN X,Y DIRECTIONS FOR MESH  .
C .              GENERATION                                         .
C .  DT..........TIME INCREMENT IN THE TRANSIENT ANALYSIS           .
C .  E1(I),E2(I).ELASTIC MODULI OF MATERIAL I                       .
C .  ELP(I)......ELEMENT FORCE VECTOR                               .
C .  ELXY(I,J)...ELEMENT NODE COORDINATES OF ELEMENT NODE I         .
C .                J=1 FOR X-COORD, J=2 FOR Y-COORD                 .
C .  G12(I),G23(I),G13(I)..SHEAR MODULI OF MATERIAL I               .
C .  GF(I).......GLOBAL FORCE VECTOR; SOLUTION VECTOR FROM 'SOLVE'  .
C .  GFO(I)......SOLUTION VECTOR AT CURRENT TIME                    .
C .  GF1(I)......FIRST TIME DERIVATIVE OF THE SOLUTION (VELOCITY)   .
C .  GF2(I)......SECOND TIME DERIVATIVE OF THE SOLUTION (ACCELERATION).
C .  GSTIF(N,M)..GLOBAL STIFFNESS MATRIX (IN BANDED FORM)           .
C .  H...........THICKNESS OF THE PLATE                             .
C .  IBDY(I).....LOCATION AND DIRECTION OF SPECIFIED GLOBAL         .
C .              DISPLACEMENTS                                      .
C .  IBSF(I).....LOCATION AND DIRECTION OF SPECIFIED NONZERO GLOBAL .
C .              FORCES                                             .
C .  IEL.........INDICATOR FOR THE ELEMENT TYPE:                    .
C .                 IEL=1, 4-NODE ELEMENT                           .
C .                 IEL=2, 8- OR 9-NODE ELEMENT                     .
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C  .  IMESH.......INDICATOR FOR MESH GENERATION                              .
C  .                  (0-READ IN, 1-SQUARE MESH IS GENERATED)                 .
C  .  ITEM........INDICATOR FOR TRANSIENT ANALYSIS (1-YES, 0-NO)             .
C  .  MTL(L)......MATERIAL NUMBER OF LAYER L                                 .
C  .  NBDY........TOTAL NUMBER OF SPECIFIED DEGREES OF FREEDOM               .
C  .  NBSF........TOTAL NUMBER OF SPECIFIED NONZERO FORCES                   .
C  .  NCMAX.......VALUE OF THE COLUMN-DIMENSION OF GSTIF                     .
C  .  NDF.........NUMBER OF DEGREES OF FREEDOM PER NODE (U,V,W,SX,SY)        .
C  .  NEM.........NUMBER OF ELEMENTS                                         .
C  .  NEQ.........TOTAL NUMBER OF DEGREES OF FREEDOM (NODESxNODAL DOF)       .
C  .  NHBW........HALF-BAND WIDTH OF GLOBAL STIFFNESS MATRIX                 .
C  .  NLAY........NUMBER OF LAYERS IN THE LAMINATE                           .
C  .  NMTL........NUMBER OF DIFFERENT MATERIALS IN THE LAMINATE              .
C  .  NN..........NUMBER OF DEGREES OF FREEDOM PER NODE                      .
C  .                  (NODES PER ELEMENT x NODAL DOF)                         .
C  .  NNM.........NUMBER OF GLOBAL NODES                                     .
C  .  NOD(I,J)....ELEMENT CONNECTIVITY MATRIX                                .
C  .  NOZERO......INDICATOR FOR ZERO(NOZERO=0) OR NONZERO(NOZERO=1)          .
C  .              INITIAL CONDITIONS FOR TRANSIENT ANALYSIS                  .
C  .  NPE.........NUMBER OF NODES PER ELEMENT (4, 8 OR 9)                    .
C  .  NPRNT.......INDICATOR FOR PRINTING ELEMENT MATRICES AND FORCE          .
C  .              VECTORS (1-PRINT, 0-DO NOT PRINT)                          .
C  .  NRMAX.......VALUE OF THE ROW-DIMENSION OF GSTIF                        .
C  .  NSTP........TIME STEP AT WHICH THE LOAD IS REMOVED FROM THE            .
C  .              PLATE (IN THE TRANSIENT ANALYSIS)                          .
C  .  NT..........CURRENT TIME STEP NUMBER IN THE TRANSIENT ANALYSIS         .
C  .  NTIME.......NUMBER OF TIME STEPS IN THE TRANSIENT ANALYSIS             .
C  .  NX..........NUMBER OF DIVISIONS ALONG X-AXIS FOR MESH GENERATION       .
C  .  NY..........NUMBER OF DIVISIONS ALONG Y-AXIS FOR MESH GENERATION       .
C  .  PO..........INTENSITY OF APPLIED TRANSVERSE UNIFORM PRESSURE           .
C  .  QBAR(I,J,L).TRANSFORMED STRESS-STRAIN MATRIX OF LAYER L                .
C  .  RHO(I)......DENSITY OF MATERIAL I                                      .
C  .  STIF(N,M)...ELEMENT STIFFNESS MATRIX                                   .
C  .  T...........TIME VARIABLE IN THE TRANSIENT ANALYSIS                    .
C  .  TH(L).......THICKNESS OF LAYER L                                       .
C  .  THETA(L)....FIBER DIRECTION ORIENTATION OF LAYER L                     .
C  .  TITLE.......TITLE FROM INPUT DATA FILE                                 .
C  .  VBDY(I).....VALUES OF THE DISPLACEMENTS CORRESPONDING TO IBDY(I)       .
C  .  VBSF(I).....VALUES OF SPECIFIED FORCES CORRESPONDING TO IBSF(I)        .
C  .  W0,W1,W2....ARRAYS CORRESPONDING TO GF0,GF1,GF2 IN AN ELEMENT          .
C  .  X(I),Y(I)...X AND Y COORDINATES OF GLOBAL NODE I                       .
C  .                                                                         .
C  ...........................................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      CHARACTER DATFILE*20,OUTPTF*20
      DIMENSION GSTIF(1000,200),GF(500),GF0(500),GF1(500),GF2(500),
     *          VBDY(400),IBDY(400),VBSF(400),IBSF(400),TITLE(20),
     *          E1(20),E2(20),G12(20),G23(20),G13(20),ANU12(20),
     *          RHO(20),MTL(20),THETA(20),TH(20),QBAR(5,5,20)
      COMMON/STF/ELXY(9,2),STIF(80,80),ELP(80),W0(80),W1(80),W2(80),
     *          A(5,5),B(3,3),D(3,3),A0,A1,A2,A3,A4,RHO1,RHO2,RHO3
      COMMON/MSH/NOD(200,9),X(225),Y(225),DX(15),DY(15)
      DATA NDF,NRMAX,NCMAX/5,1000,200/
C
      PI=3.14159265358
C
      WRITE(*,'(A27)') ' INPUT THE *.DAT FILE NAME '
```

```
      READ(*,'(A20)') DATFILE
      WRITE(*,'(A27)') ' TYPE THE OUTPUT FILE NAME '
      READ(*,'(A20)') OUTPTF
C
      IPRFIL=1
      IWRITE=2
C
      OPEN (UNIT=IPRFIL,STATUS='OLD',FORM='FORMATTED',FILE=DATFILE)
      OPEN (UNIT=IWRITE,STATUS='NEW',FORM='FORMATTED',FILE=OUTPTF)
C
C     ..........................................................
C     .             P R E P R O C E S S O R   U N I T          .
C     ..........................................................
C
C  Read title and control parameters for the program
C
      READ(1,260) TITLE
      READ(1,*) IEL,NPE,IMESH,NPRNT,ITEM,NTIME,NSTP,NOZERO
C
C  General Mesh - defined by element connectivity and nodal points
C
      IF (IMESH.EQ.0) THEN
       READ(1,*) NEM,NNM
       DO  10  I=1,NEM
   10     READ(1,*) (NOD(I,J),J=1,NPE)
       READ(1,*) (X(I),Y(I),I=1,NNM)
      END IF
C
C  Rectangular Generated Mesh - generated by defining number of element
C   subdivisions and length of subdivisions in X and Y directions
C
   20 IF (IMESH.EQ.1) THEN
       READ(1,*) NX,NY
       NX1=IEL*NX
       NY1=IEL*NY
       READ(1,*) (DX(I),I=1,NX1)
       READ(1,*) (DY(I),I=1,NY1)
       CALL MESH(IEL,NX,NY,NPE,NNM,NEM)
      END IF
C
C  Read the properties of materials used in the plate
C
   30 READ(1,*) NMTL
      DO 32 I=1,NMTL
   32   READ(1,*) E1(I),E2(I),G12(I),G13(I),G23(I),ANU12(I),RHO(I)
C
C  Read the laminate stacking sequence or lay-up
C
      READ(1,*) NLAY
      DO 34 I=1,NLAY
   34   READ(1,*) MTL(I),THETA(I),TH(I)
C
C  Read pressure, specified forces, and specified displacements
C
      READ(1,*) PO
      READ(1,*) NBDY
      READ(1,*) (IBDY(I),I=1,NBDY)
      READ(1,*) (VBDY(I),I=1,NBDY)
```

```
      READ(1,*) NBSF
      IF (NBSF.NE.0) THEN
        READ(1,*) (IBSF(I),I=1,NBSF)
        READ(1,*) (VBSF(I),I=1,NBSF)
      END IF
C
C  Transient analysis information is read
C
   35 IF (ITEM.EQ.1) THEN
        READ(1,*) DT,ALFA
C
C  Non-zero initial displacements and velocities
C
        IF (NOZERO.EQ.1) THEN
          NEQ=NNM*NDF
          READ(1,*) (GF0(I),I=1,NEQ)
          READ(1,*) (GF1(I),I=1,NEQ)
        END IF
C
C  Time Integration parameters (Newmark scheme)
C
   36   BETA=0.25*(0.5+ALFA)**2
        DT2=DT*DT
        A0=1.0/BETA/DT2
        A2=1.0/BETA/DT
        A1=ALFA*A2
        A3=0.5/BETA-1.0
        A4=ALFA/BETA-1.0
C
C  Initialize disp., vel. and accel. vectors if not specified
C
        IF(NOZERO.EQ.0) THEN
          DO 38 I=1,NEQ
            GF0(I)=0.0
            GF1(I)=0.0
            GF2(I)=0.0
   38     CONTINUE
        END IF
      END IF
C
C     ..............................................................
C     .                    P R O C E S S O R   U N I T             .
C     ..............................................................
C
C  Compute total DOF's 'NEQ', and element DOF's 'NN'
C
   40 NEQ=NNM*NDF
      NN=NPE*NDF
C
C  Compute the plate material stiffness and inertial properties
C
      CALL MATPROP(E1,E2,G12,G13,G23,ANU12,NLAY,MTL,THETA,RHO,TH,A,B,D,
     *             QBAR,H,RHO1,RHO2,RHO3)
C
C  Print the program parameters and the mesh information
C
      WRITE(2,260) TITLE
      WRITE(2,310) IEL,NPE
```

```
      WRITE(2,320) NEM,NNM,NDF
C
C  Print the material properties and stacking sequence
C
      DO 50 I=1,NMTL
   50   WRITE(2,330) I,E1(I),E2(I),G12(I),G13(I),G23(I),ANU12(I),RHO(I)
      WRITE(2,326)
      WRITE(2,327)
      DO 55 I=1,NLAY
   55   WRITE(2,328) I,MTL(I),THETA(I)*180.0/PI,TH(I)
      WRITE(2,329) H
C
C  Print the A, B, and D matrices and inertial terms
C
      WRITE(2,331)
      WRITE(2,332)
      WRITE(2,325) (A(1,J),J=1,3)
      WRITE(2,325) (A(2,J),J=1,3)
      WRITE(2,325) (A(3,J),J=1,3)
      WRITE(2,334)
      WRITE(2,325) (B(1,J),J=1,3)
      WRITE(2,325) (B(2,J),J=1,3)
      WRITE(2,325) (B(3,J),J=1,3)
      WRITE(2,336)
      WRITE(2,325) (D(1,J),J=1,3)
      WRITE(2,325) (D(2,J),J=1,3)
      WRITE(2,325) (D(3,J),J=1,3)
      WRITE(2,338)
      WRITE(2,325) A(4,4),A(4,5),A(5,5)
      WRITE(2,339)
      WRITE(2,325) RHO1,RHO2,RHO3
C
C  Print specified displacements, pressure and specified forces
C
      WRITE(2,345) NBDY
      WRITE(2,280) (IBDY(I),VBDY(I),I=1,NBDY)
      WRITE(2,342) PO
      WRITE(2,350) NBSF
      WRITE(2,280) (IBSF(I),VBSF(I),I=1,NBSF)
      WRITE(2,360)
C
C  Print element connectivity and nodal point coordinates
C
      DO 60 I=1,NEM
   60   WRITE(2,270) I,(NOD(I,J),J=1,NPE)
      WRITE(2,370)
      WRITE(2,375) (I,X(I),Y(I),I=1,NNM)
C
C  Compute the half-band width 'NHBW' of global stiffness matrix
C
      NHBW=0
      DO 70 N=1,NEM
       DO 70 I=1,NPE
         DO 70 J=1,NPE
           NW=(IABS(NOD(N,I)-NOD(N,J))+1)*NDF
           IF (NHBW.LT.NW) NHBW=NW
   70 CONTINUE
      WRITE(2,400)NHBW
```

```
C
      T=0.0
      IF (ITEM.EQ.1) WRITE(2,460) DT,ALFA,BETA,A0,A1,A2,A3,A4
C
C ------- DO-Loop on number of time steps begins here ---------------
C
      DO 220 NT=1,NTIME
       IF (ITEM.EQ.1.AND.NT.GE.NSTP) P0=0.0
C
C  Initialize the global stiffness matrix and force vector
C
       DO 80 I=1,NEQ
         GF(I)=0.0
         DO  80  J=1,NHBW
   80        GSTIF(I,J)=0.0
C
C  Convert global information to the element level
C
       DO 130 N=1,NEM
         L=0
         DO 90 I=1,NPE
           NI=NOD(N,I)
           ELXY(I,1)=X(NI)
           ELXY(I,2)=Y(NI)
           LI=(NI-1)*NDF
           DO 90 J=1,NDF
             LI=LI+1
             L=L+1
             W0(L)=GF0(LI)
             W1(L)=GF1(LI)
             W2(L)=GF2(LI)
   90     CONTINUE
C
C  Compute the element stiffness and mass matrices
C
         CALL STIFF(IEL,NPE,NN,P0,ITEM,NT,NOZERO)
         IF (NPRNT.EQ.1) THEN
           WRITE(2,380)
           DO 100 I=1,NN
  100        WRITE (2,300) (STIF(I,J),J=1,NN)
           WRITE(2,410)
           WRITE(2,300) (ELP(I),I=1,NN)
           WRITE(2,410)
         END IF
C
C  Assemble element stiffness matrices to get global stiffness matrix
C
         DO 130 I=1,NPE
           NR=(NOD(N,I)-1)*NDF
           DO 130 II=1,NDF
             NR=NR+1
             L=(I-1)*NDF+II
             GF(NR)=GF(NR)+ELP(L)
             DO 130 J=1,NPE
               NCL=(NOD(N,J)-1)*NDF
               DO 130 JJ=1,NDF
                 M=(J-1)*NDF+JJ
                 NC=NCL+JJ-NR+1
```

```
                 IF (NC.GT.0) GSTIF(NR,NC)=GSTIF(NR,NC)+STIF(L,M)
  130   CONTINUE
C
C  The global system equations are now ready for implementing the
C   force and displacement boundary conditions
C
        IF ((NBSF.GT.0).AND.(NOZERO.EQ.0.OR.ITEM.EQ.0)) THEN
          DO 140 I=1,NBSF
            NB=IBSF(I)
            GF(NB)=GF(NB)+VBSF(I)
  140     CONTINUE
        END IF
  145   CALL BNDY(NRMAX,NCMAX,NEQ,NHBW,GSTIF,GF,NBDY,IBDY,VBDY)
C
C  Call subroutine SOLVE to solve the global system of equations
C   (the solution is returned in GF(I))
C
        CALL SOLVE (NRMAX,NCMAX,NEQ,NHBW,GSTIF,GF,0)
        IF (ITEM.EQ.0) GOTO 180
C
C  Calculate the second time derivative when initial conditions
C   are non-zero
C
          IF ((NOZERO.EQ.0).OR.(NT.GT.1)) GOTO 160
            DO 150 I=1,NEQ
  150         GF2(I)=GF(I)
            GOTO 210
C
C  Calculate new velocities and accelerations
C
  160     T=T+DT
          DO 170 I=1,NEQ
            GF0(I)=A0*(GF(I)-GF0(I))-A2*GF1(I)-A3*GF2(I)
            GF1(I)=GF1(I)+DT*(1.0-ALFA)*GF2(I)+DT*ALFA*GF0(I)
            GF2(I)=GF0(I)
            GF0(I)=GF(I)
  170     CONTINUE
C
C  Print the time step and resulting generalized displacements
C
        WRITE(2,470) T
  180   WRITE(2,480)
        WRITE(2,490)(((I+4)/NDF),GF(I),GF(I+1),GF(I+2),GF(I+3),
     *               GF(I+4),I=1,NEQ,NDF)
        WRITE(2,410)
C
C     .............................................................
C     .              P O S T P R O C E S S O R   U N I T          .
C     .............................................................
C
C  Compute strains and stresses (at the Gauss points)
C
        WRITE(2,440)
        WRITE(2,450)
        DO 200 N=1,NEM
          L=0
          DO 190 I=1,NPE
            NI=NOD(N,I)
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          ELXY(I,1)=X(NI)
          ELXY(I,2)=Y(NI)
          LI=(NI-1)*NDF
          DO 190 J=1,NDF
            LI=LI+1
            L=L+1
            WO(L)=GF(LI)
  190     CONTINUE
         CALL STRESS (NPE,NDF,IEL,ELXY,WO,QBAR,NLAY,TH,H)
  200   CONTINUE
C
       IF(ITEM.EQ.0) GOTO 230
  210     WRITE(2,390)
  220 CONTINUE
C
C  ------- End of DO-Loop on the number of time steps --------------
C
  230 STOP
C
C     ..............................................................
C     .                    F  O  R  M  A  T  S                     .
C     ..............................................................
C
  260 FORMAT(20A4)
  270 FORMAT(6X,I5,2X,9I5)
  280 FORMAT((8X,3(2X,I4,2X,E12.5)))
  300 FORMAT(5(2X,E12.5))
  310 FORMAT(/,5X,'ELEMENT TYPE(1=LINEAR,2=QUADRATIC) =',I2,5X,
     *' NODES PER ELEMENT=',I2)
  320 FORMAT(10X,'ACTUAL NUMBER OF ELEMENTS IN THE MESH=',I3,
     */,10X,'NUMBER OF NODES IN THE MESH =',I3,
     */,10X,'DEGREES OF FREEDOM =',I2,/)
  325 FORMAT(7X,3(3X,E12.5))
  326 FORMAT(/,5X,'LAMINATE STACKING SEQUENCE')
  327 FORMAT(/,8X,'LAYER',3X,'MTL #',4X,'THETA',5X,'THICKNESS')
  328 FORMAT(8X,I3,6X,I2,2X,F10.5,2X,E12.5)
  329 FORMAT(/,8X,'TOTAL THICKNESS =',E12.5)
  330 FORMAT(5X,'MATERIAL ',I2,' PROPERTIES:',/,10X,'MODULUS,E1=',E12.5,
     */,10X,'MODULUS,E2=',E12.5,/,10X,'SHEAR MODULUS,G12=',E12.5,
     */,10X,'SHEAR MODULUS,G13=',E12.5,/,10X,'SHEAR MODULUS,G23=',E12.5,
     */,10X,'POISSONS RATIO,NU12=',E12.5,
     */,10X,'MATERIAL DENSITY,RHO=',E12.5,/)
  331 FORMAT(/,5X,'LAMINATE PLATE PROPERTIES')
  332 FORMAT(/,8X,'A MATRIX TERMS')
  334 FORMAT(/,8X,'B MATRIX TERMS')
  336 FORMAT(/,8X,'D MATRIX TERMS')
  338 FORMAT(/,8X,'SHEAR TERMS: A44, A45, A55')
  339 FORMAT(/,8X,'INERTIAL TERMS RHO1, RHO2, RHO3')
  342 FORMAT(/,5X,'UNIFORMLY DISTRIBUTED LOAD, P0 =',E12.5)
  345 FORMAT(/,5X,'NUMBER OF SPECIFIED DISPLACEMENTS=',I5,
     */,5X,'SPECIFED DISPLACEMENTS AND THEIR VALUES FOLLOW:')
  350 FORMAT(/,5X,'NUMBER OF SPECIFIED FORCES=',I4,/,5X,
     *'SPECIFIED FORCE DEGREES OF FREEDOM AND THEIR VALUES FOLLOW:')
  360 FORMAT(/,5X,'BOOLEAN (CONNECTIVITY) MATRIX-NOD(I,J)',/)
  370 FORMAT(/,5X,'COORDINATES OF THE GLOBAL NODES:',/)
  375 FORMAT(2(2X,I4,3X,E12.5,3X,E12.5))
  380 FORMAT(/,5X,'ELEMENT STIFFNESS AND FORCE MATRICES:',/)
  390 FORMAT(120(':'),//)
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  400 FORMAT(/,5X,'HALF BAND WIDTH OF GLOBAL STIFFNESS MATRIX = ',I5,/)
  410 FORMAT(//)
  440 FORMAT(5X,'LAMINATE STRAINS AND STRESSES AT GAUSS POINTS',/)
  450 FORMAT(1X,'(X-COORD,Y-COORD)',
     */,2X,'LOC',5X,'EPSILONX',4X,'EPSILONY',5X,'GAMMAXY',5X,
     *'KAPPAX',6X,'KAPPAY',6X,'KAPPAXY',/,2X,'LAY',4X,'Z-COORD',6X,
     *'SIGMAX',6X,'SIGMAY',7X,'TAUXY',7X,'TAUYZ',7X,'TAUXZ')
  460 FORMAT(/,5X,'DT=',E10.4,5X,'ALFA=',E10.4,5X,'BETA=',E10.4,/,10X,
     *'TEMPORAL PARAMETERS A0,A1,A2,A3,A4:',5E12.4,/)
  470 FORMAT(/,5X,'TIME=',E10.3,/)
  480 FORMAT(/,5X,'GENERALIZED DISPLACEMENTS (U,V,W,SX,SY) PER NODE',/)
  490 FORMAT(2X,I4,2X,E12.5,2X,E12.5,2X,E12.5,2X,E12.5,2X,E12.5)
      END
C
C ***********************************************************************
C *                     S U B R O U T I N E S                           *
C ***********************************************************************
C
      SUBROUTINE MATPROP(E1,E2,G12,G13,G23,ANU12,NLAY,MTL,THETA,RHO,
     *                   T,A,B,D,QBAR,H,RHO1,RHO2,RHO3)
C
C ......................................................................
C  . THIS SUBROUTINE CALCULATES THE Q AND QBAR MATRICES FOR EACH LAYER .
C  . AND THE LAMINATE MATERIAL PROPERTY MATRICES A(I,J),B(I,J),D(I,J)  .
C  . AND THE INERTIAL TERMS RHO1,RHO2,RHO3                             .
C  .                                                                   .
C  .  A(I,J)......EXTENSIONAL STIFFNESS MATRIX (I,J=1,2,3)             .
C  .  A(K,L)......SHEAR TERMS OF STIFFNESS MATRIX (K,L=4,5)            .
C  .  AK.........SHEAR CORRECTION COEFFICIENT                          .
C  .  AMM,ANN.....SINE AND COSINE OF FIBER ORIENTATION 'THETA'         .
C  .  ANU12(I),ANU21(I)...POISSON RATIOS OF MATERIAL                   .
C  .  ATOL........ZERO TOLERANCE OF STIFFNESS TERMS COMPARED OTHERS    .
C  .  B(I,J)......COUPLING STIFFNESS MATRIX (I,J=1,2,3)                .
C  .  D(I,J)......BENDING STIFFNESS MATRIX (I,J=1,2,3)                 .
C  .  E1(I),E2(I).ELASTIC MODULI OF MATERIAL I                         .
C  .  G12(I),G23(I),G13(I)..SHEAR MODULI OF MATERIAL I                 .
C  .  H...........TOTAL PLATE THICKNESS                                .
C  .  MTL(L)......MATERIAL NUMBER OF LAYER L                           .
C  .  NLAY........NUMBER OF LAYERS IN THE LAMINATE                     .
C  .  Q(I,J,L)....STRESS-STRAIN MARTIX OF LAYER L ALIGNED WITH         .
C  .                 PRINCIPAL DIRECTIONS                              .
C  .  QBAR(I,J,L).TRANSFORMED STRESS-STRAIN MATRIX OF LAYER L          .
C  .  RHO(I)......DENSITY OF MATERIAL I                                .
C  .  RHO1,RHO2,RHO3...INERTIAL PARAMETERS OF THE LAMINATE             .
C  .  TH(L).......THICKNESS OF LAYER L                                 .
C  .  THETA(L)....FIBER DIRECTION ORIENTATION OF LAYER L               .
C  .  ZBAR(L).....MID-PLANE POSITION OF LAYER L                        .
C ......................................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Q(5,5,20),QBAR(5,5,20),E1(20),E2(20),G12(20),G13(20),
     *          G23(20),ANU12(20),ANU21(20),THETA(20),T(20),
     *          MTL(20),ZBAR(20),RHO(20),A(5,5),B(3,3),D(3,3)
C
C  Shear correction factor 'AK'
C
      AK=5.0/6.0
      H=0.0
```

```
      ATOL=1.0E-09
      PI=3.141592654
C
C  This loop calculates 'Q' & 'QBAR' matrices and plate thickness 'H'
C
      DO 30 I=1,NLAY
        H=H+T(I)
        ANU21(MTL(I))=E2(MTL(I))*ANU12(MTL(I))/E1(MTL(I))
        DENOM=1.0-ANU12(MTL(I))*ANU21(MTL(I))
        Q(1,1,I)=E1(MTL(I))/DENOM
        Q(2,2,I)=E2(MTL(I))/DENOM
        Q(1,2,I)=ANU12(MTL(I))*E2(MTL(I))/DENOM
        Q(3,3,I)=G12(MTL(I))
        Q(4,4,I)=G23(MTL(I))
        Q(5,5,I)=G13(MTL(I))
C
C  Change 'THETA' to radians and calculate cos 'AMM' & sin 'ANN'
C
        THETA(I)=THETA(I)*PI/180.0
        AMM=COS(THETA(I))
        ANN=SIN(THETA(I))
C
C  Calculate lamina stiffness matrices 'QBAR' due to orientation 'THETA'
C
        QBAR(1,1,I)=Q(1,1,I)*(AMM**4.0) + 2.0*(Q(1,2,I)+2.0*Q(3,3,I))*
     *               (AMM*AMM)*(ANN*ANN) + Q(2,2,I)*(ANN**4.0)
        QBAR(1,2,I)=(Q(1,1,I)+Q(2,2,I)-4.0*(Q(3,3,I)))*(AMM*AMM)*
     *               (ANN*ANN)+ Q(1,2,I)*(AMM**4.0+ANN**4.0)
        QBAR(1,3,I)=-(Q(2,2,I)*AMM*(ANN**3.0)) + Q(1,1,I)*(AMM**3.0)*ANN
     *            - (Q(1,2,I)+2.0*Q(3,3,I))*AMM*ANN*((AMM*AMM)-(ANN*ANN))
        QBAR(2,1,I)=QBAR(1,2,I)
        QBAR(2,2,I)=Q(1,1,I)*(ANN**4.0) + 2.0*(Q(1,2,I)+2.0*Q(3,3,I))*
     *               (AMM*AMM)*(ANN*ANN) + Q(2,2,I)*(AMM**4.0)
        QBAR(2,3,I)=-(Q(2,2,I)*(AMM**3.0)*ANN) + Q(1,1,I)*AMM*(ANN**3.0)
     *            + (Q(1,2,I)+2.0*Q(3,3,I))*AMM*ANN*((AMM*AMM)-(ANN*ANN))
        QBAR(3,1,I)=QBAR(1,3,I)
        QBAR(3,2,I)=QBAR(2,3,I)
        QBAR(3,3,I)=(Q(1,1,I)+Q(2,2,I)-2.0*Q(1,2,I))*(AMM*AMM)*(ANN*ANN)
     *               + Q(3,3,I)*((AMM*AMM)-(ANN*ANN))**2.0
        QBAR(4,4,I)=Q(4,4,I)*(AMM*AMM) + Q(5,5,I)*(ANN*ANN)
        QBAR(4,5,I)=(Q(5,5,I)-Q(4,4,I))*AMM*ANN
        QBAR(5,4,I)=QBAR(4,5,I)
        QBAR(5,5,I)=Q(5,5,I)*(AMM*AMM) + Q(4,4,I)*(ANN*ANN)
   30 CONTINUE
C
C  Initialize A,B,D matrices and inertial terms
C
      DO 40 I=1,3
        DO 40 J=1,3
          A(I,J)=0.0
          B(I,J)=0.0
          D(I,J)=0.0
   40 CONTINUE
      A(4,4)=0.0
      A(4,5)=0.0
      A(5,4)=0.0
      A(5,5)=0.0
      RHO1=0.0
```

```
      RHO2=0.0
      RHO3=0.0
C
C  Calculate lamina mid-plane position 'ZBAR' from laminate mid-plane (0)
C
      ZBAR(1)=-H/2.0+T(1)/2.0
      IF(NLAY.GT.1) THEN
        DO 50 I=2,NLAY
          ZBAR(I)=ZBAR(I-1)+T(I-1)/2.0+T(I)/2.0
   50   CONTINUE
      END IF
C
C  Calculate A,B,D in-plane matrix terms
C
      DO 80 K=1,NLAY
        DO 60 I=1,3
          DO 60 J=1,3
            A(I,J)=A(I,J) + QBAR(I,J,K)*T(K)
            B(I,J)=B(I,J) + QBAR(I,J,K)*T(K)*ZBAR(K)
            D(I,J)=D(I,J) + QBAR(I,J,K)*(T(K)*ZBAR(K)**2.0
     *                    + (T(K)**3.0/12.0))
   60   CONTINUE
C
C  Calculate 'A' matrix transverse shear terms
C
        DO 70 I=4,5
          DO 70 J=4,5
            A(I,J)=A(I,J) + AK*QBAR(I,J,K)*T(K)
   70   CONTINUE
C
C  Calculate inertial terms for dynamic analysis
C
        RHO1=RHO1 + RHO(MTL(K))*T(K)
        RHO2=RHO2 + RHO(MTL(K))*T(K)*ZBAR(K)
        RHO3=RHO3 + RHO(MTL(K))*(T(K)*ZBAR(K)**2.0+(T(K)**3.0/12.0))
   80 CONTINUE
C
C  Set negligibly small laminate property terms to zero
C
      TOL=A(1,1)*ATOL
      DO 90 I=1,5
        DO 90 J=1,5
          IF (ABS(A(I,J)).LT.TOL) A(I,J)=0.0
          IF (I.LE.3.AND.J.LE.3) THEN
            IF (ABS(B(I,J)).LT.TOL) B(I,J)=0.0
            IF (ABS(D(I,J)).LT.TOL) D(I,J)=0.0
          END IF
   90 CONTINUE
C
      TOL=RHO1*ATOL
      IF (ABS(RHO2).LT.TOL) RHO2=0.0
C
      RETURN
      END
C
C***************************************************************************
C
      SUBROUTINE STIFF(IEL,NPE,NN,PO,ITEM,NT,NOZERO)
```

```
C
C ...........................................................
C . THIS SUBROUTINE IS WRITTEN FOR COMPOSITE PLATES.  THE ELEMENT IS  .
C . BASED ON A SHEAR-DEFORMABLE THEORY. HERE THE FOUR-, EIGHT- OR     .
C . NINE-NODE ISOPARAMETRIC ELEMENT WITH FIVE DEGREES OF FREEDOM      .
C . (U,V,W,SX,SY) PER NODE CAN BE USED BY SPECIFYING THE ELEMENT TYPE .
C .                                                                   .
C .                       SUBROUTINE VARIABLES                        .
C .                                                                   .
C .  A(I,J)......EXTENSIONAL STIFFNESS MATRIX (I,J=1,2,3)             .
C .  A(K,L)......SHEAR TERMS OF STIFFNESS MATRIX (K,L=4,5)            .
C .  A0,A1,A2,A3,A4...PARAMETERS IN THE TIME-APPROXIMATION SCHEME     .
C .  B(I,J)......COUPLING STIFFNESS MATRIX (I,J=1,2,3)                .
C .  CNST........INTEGRATION CONSTANT TRANSFORMED TO X,Y COORDINATES  .
C .  D(I,J)......BENDING STIFFNESS MATRIX (I,J=1,2,3)                 .
C .  DET.........DETERMINATE OF JACOBIAN TRANSFORMATION MATRIX        .
C .  ELP(I)......ELEMENT FORCE VECTOR                                 .
C .  ELXY(I,J)...ELEMENT NODE COORDINATES OF ELEMENT NODE I           .
C .               J=1 FOR X-COORD, J=2 FOR Y-COORD                    .
C .  GAUSS(I,J)..GAUSSIAN POINT COORDINATES (LOCAL VALUES)            .
C .  GDSF(J,I)...DERIVATIVE OF SHAPE FUNCTION SF(I)                   .
C .               J=1 WITH RESPECT TO X, J=2 WITH RESPECT TO Y         .
C .  H(I,J)......ELEMENT MASS MATRIX                                  .
C .  IBDY(I).....LOCATION AND DIRECTION OF SPECIFIED GLOBAL           .
C .              DISPLACEMENTS                                        .
C .  IBSF(I).....LOCATION AND DIRECTION OF SPECIFIED NONZERO GLOBAL   .
C .              FORCES                                               .
C .  IEL.........INDICATOR FOR THE ELEMENT TYPE:                      .
C .                IEL=1, 4-NODE ELEMENT                              .
C .                IEL=2, 8- OR 9-NODE ELEMENT                        .
C .  IMESH.......INDICATOR FOR MESH GENERATION                        .
C .                (0-READ IN, 1-SQUARE MESH IS GENERATED)            .
C .  ITEM........INDICATOR FOR TRANSIENT ANALYSIS (1-YES, 0-NO)       .
C .  LGP.........ORDER OF REDUCED INTEGRATION ON TRANSVERSE SHEAR TERM.
C .  NDF.........NUMBER OF DEGREES OF FREEDOM PER NODE (U,V,W,SX,SY)  .
C .  NGP.........ORDER OF NORMAL INTEGRATION ON IN-PLANE TERMS        .
C .  NN..........NUMBER OF DEGREES OF FREEDOM PER NODE                .
C .                (NODES PER ELEMENT x NODAL DOF)                    .
C .  NOZERO......INDICATOR FOR ZERO(NOZERO=0) OR NONZERO(NOZERO=1)    .
C .              INITIAL CONDITIONS FOR TRANSIENT ANALYSIS            .
C .  NPE.........NUMBER OF NODES PER ELEMENT (4, 8 OR 9)              .
C .  NT..........CURRENT TIME STEP NUMBER IN THE TRANSIENT ANALYSIS   .
C .  PO..........INTENSITY OF APPLIED TRANSVERSE UNIFORM PRESSURE     .
C .  SF(I).......VALUE OF INTERPOLATION FUNCTION OF NODE I            .
C .  STIF(I,J)...ELEMENT STIFFNESS MATRIX                             .
C .  SXX,SXY,SYY,SYX..VALUES OF SHAPE FUNCTION DERIVATIVE INTEGRALS   .
C .  SXO,SYO,SOX,SOY..VALUES OF THE PRODUCT OF SHAPE FUNCTION AND     .
C .                     SHAPE FUNCTION DERIVATIVE INTEGRALS           .
C .  SOO.........VALUE OF SHAPE FUNCTION PRODUCT INTEGRALS            .
C .    FOR Snm ABOVE X=X DERIVATIVE, Y=Y DERIVATIVE, O=SHAPE FUNCTION .
C .  RHO1,RHO2,RHO3..LAMINATE INERTIAL PROPERTIES                     .
C .  WO,W1,W2....ARRAYS CORRESPONDING TO GFO,GF1,GF2 IN AN ELEMENT    .
C .  WT(I,J).....INTEGRATION WEIGHT VALUES                            .
C .  XI,ETA......LOCAL COORDINATE VALUES OF GAUSS POINTS              .
C ...........................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/STF/ELXY(9,2),STIF(80,80),ELP(80),W0(80),W1(80),W2(80),
```

```
     *             A(5,5),B(3,3),D(3,3),A0,A1,A2,A3,A4,RHO1,RHO2,RHO3
      COMMON/SHP/SF(9),GDSF(2,9)
      DIMENSION GAUSS(4,4),WT(4,4),H(80,80)
C
C  Gaussian Point Coordinates
C
      DATA GAUSS/0.0D0, 0.0D0, 0.0D0, 0.0D0, -.57735026918962600,
     * .57735026918962600, 0.0D0, 0.0D0, -.77459666924148300, 0.0D0,
     * .77459666924148300, 0.0D0, -.86113631159405300,
     * -.33998104358485600, .33998104358485600, .86113631159405300/
C
C  Integration Weight Values
C
      DATA WT/ 2.0D0, 0.0D0, 0.0D0, 0.0D0, 1.0D0, 1.0D0, 0.0D0, 0.0D0,
     * .55555555555555600, .88888888888888900, .55555555555555600,
     *  0.0D0, .34785484513745400, .65214515486254600,
     * .65214515486254600, .34785484513745400/
C
C  Integration order of in-plane terms 'NGP', and
C    transverse shear terms (reduced-integration) 'LGP'
C
      NGP=IEL+1
      LGP=IEL
      NDF=NN/NPE
C
C  Initialize the element matrices 'STIF', 'H' and force vector 'ELP'
C
      DO 10 I=1,NN
        ELP(I)=0.0
        DO  10  J=1,NN
          H(I,J)=0.0
          STIF(I,J)=0.0
   10 CONTINUE
C
C  Gauss Quadrature (Full Integration) on in-plane terms begins here
C
      DO 80 NI=1,NGP
        DO 80 NJ=1,NGP
C
C  Convert to local Gauss point coordinates and evaluate shape function
C
          XI=GAUSS(NI,NGP)
          ETA=GAUSS(NJ,NGP)
          CALL SHAPE(NPE,XI,ETA,ELXY,DET)
          CNST=DET*WT(NI,NGP)*WT(NJ,NGP)
C
C  Distribution of constant pressure to nodal points
C
          DO 30 I=1,NPE
            L=(I-1)*NDF+3
            ELP(L)=ELP(L)+CNST*SF(I)*P0
   30     CONTINUE
C
C  Compute Stiffness matrix 'STIF' and Mass matrix 'H' coefficients
C
          II=1
          DO 70 I=1,NPE
            JJ=1
```

```
          DO 60 J=1,NPE
C
C  Integrals of the shape functions and derivatives
C
            SXX=GDSF(1,I)*GDSF(1,J)*CNST
            SYY=GDSF(2,I)*GDSF(2,J)*CNST
            SXY=GDSF(1,I)*GDSF(2,J)*CNST
            SYX=GDSF(2,I)*GDSF(1,J)*CNST
            S00=SF(I)*SF(J)*CNST
C
C  Full-Integration on in-plane stiffness terms
C
        STIF(II,JJ)=STIF(II,JJ)+A(1,1)*SXX+A(1,3)*(SXY+SYX)+A(3,3)*SYY
        STIF(II,JJ+1)=STIF(II,JJ+1)+A(1,2)*SXY+A(1,3)*SXX+A(2,3)*SYY
     *                +A(3,3)*SYX
        STIF(II+1,JJ)=STIF(II+1,JJ)+A(1,2)*SYX+A(1,3)*SXX+A(2,3)*SYY
     *                +A(3,3)*SXY
        STIF(II,JJ+3)=STIF(II,JJ+3)+B(1,1)*SXX+B(1,3)*(SXY+SYX)
     *                +B(3,3)*SYY
        STIF(II+3,JJ)=STIF(II+3,JJ)+B(1,1)*SXX+B(1,3)*(SYX+SXY)
     *                +B(3,3)*SYY
        STIF(II,JJ+4)=STIF(II,JJ+4)+B(1,2)*SXY+B(1,3)*SXX+B(2,3)*SYY
     *                +B(3,3)*SYX
        STIF(II+4,JJ)=STIF(II+4,JJ)+B(1,2)*SYX+B(1,3)*SXX+B(2,3)*SYY
     *                +B(3,3)*SXY
        STIF(II+1,JJ+1)=STIF(II+1,JJ+1)+A(2,2)*SYY+A(2,3)*(SXY+SYX)
     *                  +A(3,3)*SXX
        STIF(II+1,JJ+3)=STIF(II+1,JJ+3)+B(1,2)*SYX+B(2,3)*SYY
     *                  +B(1,3)*SXX+B(3,3)*SXY
        STIF(II+3,JJ+1)=STIF(II+3,JJ+1)+B(1,2)*SXY+B(2,3)*SYY
     *                  +B(1,3)*SXX+B(3,3)*SYX
        STIF(II+1,JJ+4)=STIF(II+1,JJ+4)+B(2,2)*SYY+B(2,3)*(SXY+SYX)
     *                  +B(3,3)*SXX
        STIF(II+4,JJ+1)=STIF(II+4,JJ+1)+B(2,2)*SYY+B(2,3)*(SYX+SXY)
     *                  +B(3,3)*SXX
        STIF(II+3,JJ+3)=STIF(II+3,JJ+3)+D(1,1)*SXX+D(1,3)*(SXY+SYX)
     *                  +D(3,3)*SYY
        STIF(II+3,JJ+4)=STIF(II+3,JJ+4)+D(1,2)*SXY+D(1,3)*SXX
     *                  +D(2,3)*SYY+D(3,3)*SYX
        STIF(II+4,JJ+3)=STIF(II+4,JJ+3)+D(1,2)*SYX+D(1,3)*SXX
     *                  +D(2,3)*SYY+D(3,3)*SXY
        STIF(II+4,JJ+4)=STIF(II+4,JJ+4)+D(2,3)*(SXY+SYX)+D(3,3)*SXX
     *                  +D(2,2)*SYY
C
C  Mass matrix terms 'H' for transient analysis
C
            IF (ITEM.EQ.1) THEN
             H(II,JJ)=H(II,JJ)+RHO1*S00
             H(II,JJ+3)=H(II,JJ+3)+RHO2*S00
             H(II+3,JJ)=H(II+3,JJ)+RHO2*S00
             H(II+1,JJ+1)=H(II+1,JJ+1)+RHO1*S00
             H(II+1,JJ+4)=H(II+1,JJ+4)+RHO2*S00
             H(II+4,JJ+1)=H(II+4,JJ+1)+RHO2*S00
             H(II+2,JJ+2)=H(II+2,JJ+2)+RHO1*S00
             H(II+3,JJ+3)=H(II+3,JJ+3)+RHO3*S00
             H(II+4,JJ+4)=H(II+4,JJ+4)+RHO3*S00
            END IF
C
```

```
            JJ=NDF*J+1
   60       CONTINUE
           II=NDF*I+1
   70     CONTINUE
   80 CONTINUE
C
C   Gauss Quadrature (Reduced Integration) on transverse shear terms
C
      DO 110 NI=1,LGP
        DO 110 NJ=1,LGP
          XI=GAUSS(NI,LGP)
          ETA=GAUSS(NJ,LGP)
          CALL SHAPE(NPE,XI,ETA,ELXY,DET)
          CNST=DET*WT(NI,LGP)*WT(NJ,LGP)
          II=1
C
          DO 100 I=1,NPE
            JJ=1
            DO 90 J=1,NPE
C
C   Integrals of the shape functions and derivatives
C
              SXX=GDSF(1,I)*GDSF(1,J)*CNST
              SYY=GDSF(2,I)*GDSF(2,J)*CNST
              SXY=GDSF(1,I)*GDSF(2,J)*CNST
              SYX=GDSF(2,I)*GDSF(1,J)*CNST
              SX0=GDSF(1,I)*SF(J)*CNST
              S0X=SF(I)*GDSF(1,J)*CNST
              SY0=GDSF(2,I)*SF(J)*CNST
              S0Y=SF(I)*GDSF(2,J)*CNST
              S00=SF(I)*SF(J)*CNST
C
C   Reduced-Integration on in-plane stiffness terms
C
        STIF(II+2,JJ+2)=STIF(II+2,JJ+2)+A(5,5)*SXX+A(4,5)*(SXY+SYX)
     *                       +A(4,4)*SYY
        STIF(II+2,JJ+3)=STIF(II+2,JJ+3)+A(5,5)*SX0+A(4,5)*SY0
        STIF(II+3,JJ+2)=STIF(II+3,JJ+2)+A(5,5)*S0X+A(4,5)*S0Y
        STIF(II+2,JJ+4)=STIF(II+2,JJ+4)+A(4,5)*SX0+A(4,4)*SY0
        STIF(II+4,JJ+2)=STIF(II+4,JJ+2)+A(4,5)*S0X+A(4,4)*S0Y
        STIF(II+3,JJ+3)=STIF(II+3,JJ+3)+A(5,5)*S00
        STIF(II+3,JJ+4)=STIF(II+3,JJ+4)+A(4,5)*S00
        STIF(II+4,JJ+3)=STIF(II+4,JJ+3)+A(4,5)*S00
        STIF(II+4,JJ+4)=STIF(II+4,JJ+4)+A(4,4)*S00
C
              JJ=NDF*J+1
   90       CONTINUE
           II=NDF*I+1
  100     CONTINUE
  110 CONTINUE
C
C   Element calculations for transient analysis brgin here
C
      IF (ITEM.EQ.0) RETURN
      IF (NOZERO.EQ.1.AND.NT.EQ.1) THEN
        DO 120 I=1,NN
          ELP(I)=0.0
          DO 120 J=1,NN
```

```
            ELP(I)=ELP(I)-STIF(I,J)*W0(J)
            STIF(I,J)=H(I,J)
  120     CONTINUE
          RETURN
        END IF
C
  130 DO 140 I=1,NN
        DO 140 J=1,NN
          ELP(I)=ELP(I)+H(I,J)*(A0*W0(J)+A2*W1(J)+A3*W2(J))
          STIF(I,J)=STIF(I,J)+A0*H(I,J)
  140 CONTINUE
      RETURN
      END
C
C**********************************************************************
C
        SUBROUTINE STRESS (NPE,NDF,IEL,ELXY,W,QBAR,NLAY,TH,H)
C
C  ....................................................................
C  . THIS ROUTINE EVALUATES THE STRESSES AND STRAINS AT THE GAUSS      .
C  . POINTS USING THE REDUCED INTEGRATION.                             .
C  .                                                                   .
C  .  AKAPX,AKAPY,AKAPXY..CURVATURES AT CURRENT GAUSS POINT            .
C  .  ELXY(I,J)...ELEMENT NODE COORDINATES OF ELEMENT NODE I           .
C  .               J=1 FOR X-COORD, J=2 FOR Y-COORD                    .
C  .  EPN(I)......VECTOR OF CURRENT Z-POSITION STRAINS                 .
C  .               1-EPNxx,2-EPNyy,3-GAMxy,4-GAMyz,5-GAMxz             .
C  .  EPNX0,EPNY0,GAMXY0..MID-PLANE STRAINS AT CURRENT GAUSS POINT     .
C  .  EPNX1,EPNY1,GAMXY1..LAMINATE BOTTOM STAINS AT CURRENT GAUSS POINT.
C  .  EPNX2,EPNY2,GAMXY2..LAMINATE TOP STAINS AT CURRENT GAUSS POINT   .
C  .  GAMYZ,GAMXZ..TRANSVERSE SHEAR STRAINS AT CURRENT GAUSS POINT     .
C  .  GAUSS(I,J)..GAUSSIAN POINT COORDINATES (LOCAL VALUES)            .
C  .  GDSF(J,I)...DERIVATIVE OF SHAPE FUNCTION SF(I)                   .
C  .               J=1 WITH RESPECT TO X, J=2 WITH RESPECT TO Y        .
C  .  H...........TOTAL PLATE THICKNESS                                .
C  .  IEL.........INDICATOR FOR THE ELEMENT TYPE:                      .
C  .               IEL=1, 4-NODE ELEMENT                               .
C  .               IEL=2, 8- OR 9-NODE ELEMENT                         .
C  .  L...........POINTER TO FIRST DOF OF NODE WITH 'NDF' DOF PER NODE .
C  .  NDF.........NUMBER OF DEGREES OF FREEDOM PER NODE (U,V,W,SX,SY)  .
C  .  NGP.........ORDER OF REDUCED INTEGRATION FOR STRAINS             .
C  .  NLAY........NUMBER OF PLATE LAYERS                               .
C  .  NPE.........NUMBER OF NODES PER ELEMENT (4, 8 OR 9)              .
C  .  QBAR(I,J,L).TRANSFORMED STRESS-STRAIN MATRIX OF LAYER L          .
C  .  SIGMA(I)....VECTOR OF CURRENT Z-POSITION STRESSES                .
C  .               1-Sxx,2-Syy,3-Txy,4-Tyz,5-Txz                       .
C  .  SF(I).......VALUE OF INTERPOLATION FUNCTION OF NODE I            .
C  .  TH(L).......THICKNESS OF LAYER L                                 .
C  .  W(I)........VALUES OF ELEMENT GENERALIZED DISPLACEMENTS          .
C  .  XI,ETA......LOCAL COORDINATE VALUES OF GAUSS POINTS              .
C  .  X,Y.........GLOBAL COORDINATES OF CURRENT GAUSS POINT            .
C  .  Z(L)........Z-COORDINATE OF LAYER INTERFACES (LAYER L/2)         .
C  .               L=ODD-BOTTOM OF LAYER, L=EVEN-TOP OF LAYER          .
C  ....................................................................
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/SHP/SF(9),GDSF(2,9)
      DIMENSION GAUSS(4,4),ELXY(9,2),W(80),QBAR(5,5,20),TH(20),Z(40),
```

```
     *          EPN(5),SIGMA(5)
C
C  Gaussian point coordinates
C
      DATA GAUSS/0.0D0, 0.0D0, 0.0D0, 0.0D0, -.577350269189626D0,
     * .577350269189626D0, 0.0D0, 0.0D0, -.774596669241483D0, 0.0D0,
     * .774596669241483D0, 0.0D0, -.861136311594053D0,
     * -.339981043584856D0, .339981043584856D0, .861136311594053D0/
C
C  Order of integration (reduced)  'NGP'
C
      NGP=IEL
      DO 60 NI=1,NGP
        DO 60 NJ=1,NGP
          XI=GAUSS(NI,NGP)
          ETA=GAUSS(NJ,NGP)
          CALL SHAPE (NPE,XI,ETA,ELXY,DET)
          EPNX0=0.0
          EPNY0=0.0
          GAMXY0=0.0
          GAMYZ=0.0
          GAMXZ=0.0
          AKAPX=0.0
          AKAPY=0.0
          AKAPXY=0.0
          X=0.0
          Y=0.0
          DO 20 I=1,NPE
            L=(I-1)*NDF+1
            X=X+SF(I)*ELXY(I,1)
            Y=Y+SF(I)*ELXY(I,2)
C
C  Compute the midplane strains and curvatures and average
C    tranverse shear strain
C
   10        EPNX0=EPNX0+GDSF(1,I)*W(L)
            EPNY0=EPNY0+GDSF(2,I)*W(L+1)
            GAMXY0=GAMXY0+GDSF(2,I)*W(L)+GDSF(1,I)*W(L+1)
            GAMYZ=GAMYZ+SF(I)*W(L+4)+GDSF(2,I)*W(L+2)
            GAMXZ=GAMXZ+SF(I)*W(L+3)+GDSF(1,I)*W(L+2)
            AKAPX=AKAPX+GDSF(1,I)*W(L+3)
            AKAPY=AKAPY+GDSF(2,I)*W(L+4)
            AKAPXY=AKAPXY+GDSF(2,I)*W(L+3)+GDSF(1,I)*W(L+4)
   20      CONTINUE
C
C  Compute strains at the bottom (1) and top (2) of the laminate
C
          EPNX1=EPNX0-(H/2.0)*AKAPX
          EPNY1=EPNY0-(H/2.0)*AKAPY
          GAMXY1=GAMXY0-(H/2.0)*AKAPXY
          EPNX2=EPNX0+(H/2.0)*AKAPX
          EPNY2=EPNY0+(H/2.0)*AKAPY
          GAMXY2=GAMXY0+(H/2.0)*AKAPXY
C
C  Print midplane strains and curvatures, and surface strains
C
          WRITE(2,100) X,Y
          WRITE(2,110) EPNX0,EPNY0,GAMXY0,AKAPX,AKAPY,AKAPXY
```

```
        WRITE(2,120) EPNX1,EPNY1,GAMXY1
        WRITE(2,130) EPNX2,EPNY2,GAMXY2
C
C  Compute z-coordinate at the lamina interfaces
C
        Z(0)=-H/2.0
        DO 50 LL=1,NLAY
          Z(2*LL-1)=Z(2*LL-2)
          Z(2*LL)=Z(2*LL-1)+TH(LL)
C
C  Compute layer strains at layer interfaces
C
          DO 40 KK=1,2
            EPN(1)=EPNX0+Z(2*(LL-1)+KK)*AKAPX
            EPN(2)=EPNY0+Z(2*(LL-1)+KK)*AKAPY
            EPN(3)=GAMXY0+Z(2*(LL-1)+KK)*AKAPXY
            EPN(4)=GAMYZ
            EPN(5)=GAMXZ
C
C  This loop computes the layer stresses at layer interfaces
C
            DO 30 II=1,5
              SIGMA(II)=0.0
              DO 30 JJ=1,5
                SIGMA(II)=SIGMA(II)+QBAR(II,JJ,LL)*EPN(JJ)
   30         CONTINUE
C
C  Print the stresses at the lamina interfaces
C
            WRITE(2,140) LL,Z(2*(LL-1)+KK),(SIGMA(MM),MM=1,5)
   40       CONTINUE
   50     CONTINUE
   60 CONTINUE
   RETURN
C
C
  100 FORMAT(/,1X,'(',E12.4,',',E12.4,')')
  110 FORMAT(2X,'MID',1X,(1X,3E12.4))
  120 FORMAT(2X,'BOT',1X,(1X,3E12.4))
  130 FORMAT(2X,'TOP',1X,(1X,3E12.4))
  140 FORMAT(2X,I2,1X,E12.4,1X,5E12.4)
      END
C
C*************************************************************************
C
      SUBROUTINE BNDY(NRMAX,NCMAX,NEQ,NHBW,S,SL,NBDY,IBDY,VBDY)
C
C  ..................................................................
C  . SUBROUTINE USED TO IMPOSE BOUNDARY CONDITIONS ON BANDED EQUATIONS .
C  .                                                                .
C  .  IBDY(I).....LOCATION AND DIRECTION OF SPECIFIED GLOBAL        .
C  .              DISPLACEMENTS                                     .
C  .  NBDY........TOTAL NUMBER OF SPECIFIED GLOBAL DISPLACEMENTS    .
C  .  NCMAX.......VALUE OF THE COLUMN-DIMENSION OF S                .
C  .  NEQ.........TOTAL NUMBER OF DEGREES OF FREEDOM (NODESxNODAL DOF) .
C  .  NHBW........HALF-BAND WIDTH OF GLOBAL STIFFNESS MATRIX        .
C  .  NRMAX.......VALUE OF THE ROW-DIMENSION OF S                   .
C  .  S(M,N)......GLOBAL STIFFNESS MATRIX (IN BANDED FORM)          .
```

```
C  .  SL(M).......GLOBAL FORCE VECTOR                                    .
C  .  SVAL........VALUE OF CURRENT SPECIFIED DISPLACEMENT                 .
C  .  VBDY(I).....VALUES OF THE DISPLACEMENTS CORRESPONDING TO IBDY(I) .
C  .......................................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION S(NRMAX,NCMAX),SL(NRMAX),IBDY(NBDY),VBDY(NBDY)
C
      DO 300 NB=1,NBDY
        IE=IBDY(NB)
        SVAL=VBDY(NB)
        IT=NHBW-1
        I=IE-NHBW
        DO 100 II=1,IT
          I=I+1
          IF (I.GE.1) THEN
            J=IE-I+1
            SL(I)=SL(I)-S(I,J)*SVAL
            S(I,J)=0.0
          END IF
  100   CONTINUE
        S(IE,1)=1.0
        SL(IE)=SVAL
        I=IE
        DO 200 II=2,NHBW
          I=I+1
          IF (I.LE.NEQ) THEN
            SL(I)=SL(I)-S(IE,II)*SVAL
            S(IE,II)=0.0
          END IF
  200   CONTINUE
  300 CONTINUE
      RETURN
      END
C
C************************************************************************
C
      SUBROUTINE SOLVE(NRM,NCM,NEQNS,NBW,BAND,RHS,IRES)
C
C  .......................................................................
C  . SOLVING A BANDED SYMMETRIC SYSTEM OF EQUATIONS                       .
C  . IN RESOLVING, IRES .GT. 0, LHS ELIMINATION IS SKIPPED                .
C  .                                                                      .
C  .  BAND(M,N)...GLOBAL STIFFNESS MATRIX (IN BANDED FORM)                .
C  .  IRES........IF IRES .GT. 0 THEN FORWARD ELIMINATION IS SKIPPED      .
C  .  NBW.........HALF-BAND WIDTH OF GLOBAL STIFFNESS MATRIX              .
C  .  NCM.........VALUE OF THE COLUMN-DIMENSION OF S                      .
C  .  NEQNS.......NUMBER OF EQUATIONS (TOTAL DEGREES OF FREEDOM)          .
C  .  NRM.........VALUE OF THE ROW-DIMENSION OF S                         .
C  .  RHS(M)......GLOBAL FORCE VECTOR                                     .
C  .  SVAL........VALUE OF CURRENT SPECIFIED DISPLACEMENT                 .
C  .  VBDY(I).....VALUES OF THE DISPLACEMENTS CORRESPONDING TO IBDY(I) .
C  .......................................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION BAND(NRM,NCM),RHS(NRM)
C
      MEQNS=NEQNS-1
```

```
C
C  Forward Elimination
C
      IF (IRES.EQ.0) THEN
        DO 500 NPIV=1,MEQNS
          NPIVOT=NPIV+1
          LSTSUB=NPIV+NBW-1
          IF (LSTSUB.GT.NEQNS) LSTSUB=NEQNS
          DO 400 NROW=NPIVOT,LSTSUB
C
C  Invert rows and columns for row factor
C
            NCOL=NROW-NPIV+1
            FACTOR=BAND(NPIV,NCOL)/BAND(NPIV,1)
            DO 200 NCOL=NROW,LSTSUB
              ICOL=NCOL-NROW+1
              JCOL=NCOL-NPIV+1
              BAND(NROW,ICOL)=BAND(NROW,ICOL)-FACTOR*BAND(NPIV,JCOL)
  200       CONTINUE
            RHS(NROW)=RHS(NROW)-FACTOR*RHS(NPIV)
  400     CONTINUE
  500   CONTINUE
      END IF
C
C  Skip forward elimination of matrix for IRES .GT. 0
C
      IF (IRES.GT.0) THEN
   90   DO 100 NPIV=1,MEQNS
          NPIVOT=NPIV+1
          LSTSUB=NPIV+NBW-1
          IF(LSTSUB.GT.NEQNS) LSTSUB=NEQNS
          DO 110 NROW=NPIVOT,LSTSUB
            NCOL=NROW-NPIV+1
            FACTOR=BAND(NPIV,NCOL)/BAND(NPIV,1)
            RHS(NROW)=RHS(NROW)-FACTOR*RHS(NPIV)
  110     CONTINUE
  100   CONTINUE
      END IF
C
C  Back Substitution
C
      DO 800 IJK=2,NEQNS
        NPIV=NEQNS-IJK+2
        RHS(NPIV)=RHS(NPIV)/BAND(NPIV,1)
        LSTSUB=NPIV-NBW+1
        IF (LSTSUB.LT.1) LSTSUB=1
        NPIVOT=NPIV-1
        DO 700 JKI=LSTSUB,NPIVOT
          NROW=NPIVOT-JKI+LSTSUB
          NCOL=NPIV-NROW+1
          FACTOR=BAND(NROW,NCOL)
          RHS(NROW)=RHS(NROW)-FACTOR*RHS(NPIV)
  700   CONTINUE
  800 CONTINUE
      RHS(1)=RHS(1)/BAND(1,1)
      RETURN
      END
C
```

```
C************************************************************************
C
      SUBROUTINE SHAPE(NPE,XI,ETA,ELXY,DET)
C
C   ..............................................................
C   . THIS SUBROUTINE EVALUATES THE INTERPOLATION FUNCTIONS (SF(I)) AND .
C   . ITS DERIVATIVES WITH RESPECT TO NATURAL COORDINATES (DSF(I,J)),   .
C   . AND THE DERIVATIVES OF SF(I) WITH RESPECT TO GLOBAL COORDINATES   .
C   . FOR 4, 8, AND 9-NODED RECTANGULAR ISOPARAMETIRC ELEMENTS          .
C   .                                                                   .
C   .  DET.........DETERMINATE OF JACOBIAN TRANSFORMATION MATRIX        .
C   .  DSF(I,J)....LOCAL DERIVATIVE OF SF(J) WITH RESPECT TO XI IF I=1  .
C   .                AND WITH RESPECT TO ETA IF I=2.                    .
C   .  ELXY(I,J)...ELEMENT NODE COORDINATES OF ELEMENT NODE I           .
C   .                J=1 FOR X-COORD, J=2 FOR Y-COORD                   .
C   .  GDSF(I,J)...GLOBAL DERIVATIVE OF SF(J) WITH RESPECT TO X IF I=1  .
C   .                AND WITH RESPECT TO Y IF I=2.                      .
C   .  GJ(I,J).....JACOBIAN MATRIX                                      .
C   .  GIINV(I,J)..INVERSE OF THE JACOBIAN MATRIX                       .
C   .  NP(I).......ARRAY OF ELEMENT NODES (USED FOR DEFINING SF AND DSF).
C   .  NPE.........NUMBER OF NODES PER ELEMENT (4, 8 OR 9)              .
C   .  SF(I).......INTERPOLATION FUNCTION FOR NODE I OF THE ELEMENT     .
C   .  XI,ETA......LOCAL COORDINATE VALUES OF GAUSS POINTS              .
C   .  XNODE(I,J)..LOCAL COORDINATES OF NODE I OF THE ELEMENT           .
C   .                J=1 FOR XI-COORD, J=2 FOR ETA-COORD                .
C   ..............................................................
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/SHP/SF(9),GDSF(2,9)
      DIMENSION ELXY(9,2),XNODE(9,2),NP(9),DSF(2,9),GJ(2,2),GJINV(2,2)
C
C  Local nodal point coordinates 'XNODE' and node numbers 'NP'
C
      DATA XNODE/-1.0D0,2*1.0D0,-1.0D0,0.0D0,1.0D0,0.0D0,-1.0D0,0.0D0,
     *           2*-1.0D0,2*1.0D0,-1.0D0,0.0D0,1.0D0,2*0.0D0/
      DATA NP/1,2,3,4,5,7,6,8,9/
C
C  Multiplication function for real variables
C
      FNC(A,B)=A*B
C
      IF (NPE-8) 60,10,80
C
C  Quadratic interpolation functions (for the EIGHT-NODE element)
C
   10 DO 40 I=1,NPE
        NI=NP(I)
        XP=XNODE(NI,1)
        YP=XNODE(NI,2)
        XI0=1.0+XI*XP
        ETA0=1.0+ETA*YP
        XI1=1.0-XI*XI
        ETA1=1.0-ETA*ETA
C
        IF (I.GT.4) GOTO 20
          SF(NI)=0.25*FNC(XI0,ETA0)*(XI*XP+ETA*YP-1.0)
          DSF(1,NI)=0.25*FNC(ETA0,XP)*(2.0*XI*XP+ETA*YP)
          DSF(2,NI)=0.25*FNC(XI0,YP)*(2.0*ETA*YP+XI*XP)
```

```
      GOTO 40
C
 20   IF (I.GT.6) GOTO 30
      SF(NI)=0.5*FNC(XI1,ETA0)
      DSF(1,NI)=-FNC(XI,ETA0)
      DSF(2,NI)=0.5*FNC(YP,XI1)
      GOTO 40
C
 30   SF(NI)=0.5*FNC(ETA1,XI0)
     DSF(1,NI)=0.5*FNC(XP,ETA1)
     DSF(2,NI)=-FNC(ETA,XI0)
 40 CONTINUE
    GOTO 130
C
C  Linear interpolation functions (for the FOUR-NODE element)
C
 60 DO 70 I=1,NPE
     XP=XNODE(I,1)
     YP=XNODE(I,2)
     XI0=1.0+XI*XP
     ETA0=1.0+ETA*YP
     SF(I)=0.25*FNC(XI0,ETA0)
     DSF(1,I)=0.25*FNC(XP,ETA0)
     DSF(2,I)=0.25*FNC(YP,XI0)
 70 CONTINUE
    GOTO 130
C
C  Quadratic interpolation functions (for the NINE-NODE element)
C
 80 DO 120 I=1,NPE
     NI=NP(I)
     XP=XNODE(NI,1)
     YP=XNODE(NI,2)
     XI0=1.0+XI*XP
     ETA0=1.0+ETA*YP
     XI1=1.0-XI*XI
     ETA1=1.0-ETA*ETA
     XI2=XP*XI
     ETA2=YP*ETA
     IF (I.GT.4) GOTO 90
      SF(NI)=0.25*FNC(XI0,ETA0)*XI2*ETA2
      DSF(1,NI)=0.25*XP*FNC(ETA2,ETA0)*(1.0+2.0*XI2)
      DSF(2,NI)=0.25*YP*FNC(XI2,XI0)*(1.0+2.0*ETA2)
      GOTO 120
 90   IF (I.GT.6) GOTO 100
      SF(NI)=0.5*FNC(XI1,ETA0)*ETA2
      DSF(1,NI)=-XI*FNC(ETA2,ETA0)
      DSF(2,NI)=0.5*FNC(XI1,YP)*(1.0+2.0*ETA2)
      GOTO 120
100   IF (I.GT.8) GOTO 110
      SF(NI)=0.5*FNC(ETA1,XI0)*XI2
      DSF(2,NI)=-ETA*FNC(XI2,XI0)
      DSF(1,NI)=0.5*FNC(ETA1,XP)*(1.0+2.0*XI2)
      GOTO 120
110   SF(NI)=FNC(XI1,ETA1)
     DSF(1,NI)=-2.0*XI*ETA1
     DSF(2,NI)=-2.0*ETA*XI1
120 CONTINUE
```

```
C
C  Transform derivatives from local (XI,ETA) to global (X,Y) derivatives
C
  130 DO 140 I=1,2
        DO 140 J=1,2
          GJ(I,J)=0.0
          DO 140 K=1,NPE
            GJ(I,J)=GJ(I,J)+DSF(I,K)*ELXY(K,J)
  140 CONTINUE
      DET=GJ(1,1)*GJ(2,2)-GJ(1,2)*GJ(2,1)
      GJINV(1,1)=GJ(2,2)/DET
      GJINV(2,2)=GJ(1,1)/DET
      GJINV(1,2)=-GJ(1,2)/DET
      GJINV(2,1)=-GJ(2,1)/DET
      DO 150 I=1,2
        DO 150 J=1,NPE
          GDSF(I,J)=0.0
          DO 150 K=1,2
            GDSF(I,J)=GDSF(I,J)+GJINV(I,K)*DSF(K,J)
  150 CONTINUE
C
      RETURN
      END
C
C***************************************************************************
C
      SUBROUTINE MESH(IEL,NX,NY,NPE,NNM,NEM)
C
C  ...............................................................
C  . THIS SUBROUTINE GENERATES ARRAY NOD(I,J) COORDINATES X(I),Y(I)    .
C  . AND MESH INFORMATION (NNM,NEM,NPE) FOR RECTANGULAR DOMAINS. THE   .
C  . DOMAIN IS DIVIDED INTO LINEAR OR QUADRATIC QUADRILATERAL ELEMENTS .
C  . (NX BY NY NONUNIFORM MESH IN GENERAL).                            .
C  .                                                                   .
C  .  DX(I),DY(I).DISTANCE BETWEEN NODES IN X,Y DIRECTIONS FOR MESH    .
C  .               GENERATION                                          .
C  .  IEL........ELEMENT TYPE (IEL=1: 4 NODES, IEL=2: 8 OR 9 NODES)    .
C  .  NEM........TOTAL NUMBER OF ELEMENTS                              .
C  .  NNM........TOTAL NUMBER OF NODES                                 .
C  .  NOD(I,J)...CONNECTIVITY MATRIX                                   .
C  .  NPE........NUMBER OF NODES PER ELEMENT                           .
C  .  NX,NY......NUMBER OF ELEMENTS ALONG X,Y-DIRECTIONS               .
C  .  NXX,NYY....NUMBER OF SUBDIVISIONS BETWEEN NODES IN X,Y-DIRECTIONS.
C  .  NXX1,NYY1..NUMBER OF NODES ALONG X,Y-DIRECTIONS                  .
C  .  NYY........NUMBER OF DIVISIONS BETWEEN NODES IN Y-DIRECTION      .
C  .  X(I),Y(I)..COORDINATES OF THE ITH NODE                           .
C  ...............................................................
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/MSH/NOD(200,9),X(225),Y(225),DX(15),DY(15)
C
C  Mesh of Quadrilateral Elements with Four, Eight, or Nine nodes
C
  100 NEX1=NX+1
      NEY1=NY+1
      NXX=IEL*NX
      NYY=IEL*NY
      NXX1=NXX+1
```

```
      NYY1=NYY+1
      NEM=NX*NY
      NNM=NXX1*NYY1-(IEL-1)*NX*NY
      KO=0
      IF (NPE.EQ.9) THEN
        NNM=NXX1*NYY1
        KO=1
      END IF
C
C  Generate element connectivity array 'NOD(I,J)' of first element
C
      NOD(1,1)=1
      NOD(1,2)=IEL+1
      NOD(1,3)=NXX1+(IEL-1)*NEX1+IEL+1
      IF (NPE.EQ.9) NOD(1,3)=4*NX+5
      NOD(1,4)=NOD(1,3)-IEL
      IF (NPE.GT.4) THEN
        NOD(1,5)=2
        NOD(1,6)=NXX1+(NPE-6)
        NOD(1,7)=NOD(1,3)-1
        NOD(1,8)=NXX1+1
        IF (NPE.EQ.9) NOD(1,9)=NXX1+2
      END IF
C
C  For more than 1 element in the y-direction
C
  200 IF (NY.GT.1) THEN
        M=1
        DO 220 N=2,NY
          L=(N-1)*NX+1
          DO 210 I=1,NPE
  210       NOD(L,I)=NOD(M,I)+NXX1+(IEL-1)*NEX1+KO*NX
          M=L
  220   CONTINUE
      END IF
C
C  For more than 1 element in the x-direction
C
  230 IF (NX.GT.1) THEN
        DO 260 NI=2,NX
          DO 240 I=1,NPE
            K1=IEL
            IF(I.EQ.6.OR.I.EQ.8)K1=1+KO
  240       NOD(NI,I)=NOD(NI-1,I)+K1
            M=NI
            DO 260 NJ=2,NY
              L=(NJ-1)*NX+NI
              DO 250 J=1,NPE
  250         NOD(L,J)=NOD(M,J)+NXX1+(IEL-1)*NEX1+KO*NX
              M=L
  260   CONTINUE
      END IF
C
C  Generate the nodal coorinates arrays 'X(I)' and 'Y(I)'
C
  270 YC=0.0
C
C  For 4 or 8-noded elements
```

```
C
      IF (NPE.EQ.9) GOTO 310
        DO 300 NI=1,NEY1
          I=(NXX1+(IEL-1)*NEX1)*(NI-1)+1
          J=(NI-1)*IEL+1
          X(I)=0.0
          Y(I)=YC
          DO 280 NJ=1,NXX
            I=I+1
            X(I)=X(I-1)+DX(NJ)
            Y(I)=YC
  280     CONTINUE
C
C  For 8-noded elements
C
          IF (NI.GT.NY.OR.IEL.EQ.1) GOTO 300
            J=J+1
            YC=YC+DY(J-1)
            I=I+1
            X(I)=0.0
            Y(I)=YC
            DO 290 II=1,NX
              K=2*II-1
              I=I+1
              X(I)=X(I-1)+DX(K)+DX(K+1)
              Y(I)=YC
  290       CONTINUE
  300     YC=YC+DY(J)
        RETURN
C
C  For 9-noded elements
C
  310 DO 330 NI=1,NYY1
        I=NXX1*(NI-1)
        XC=0.0
        DO 320 NJ=1,NXX1
          I=I+1
          X(I)=XC
          Y(I)=YC
          XC=XC+DX(NJ)
  320   CONTINUE
        YC=YC+DY(NI)
  330 CONTINUE
C
      RETURN
      END
```

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